Positive and Monotone Systems

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2. Monotone Systems
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A continuous linear time-invariant system

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &=Cx(t) + Du(t),
\end{aligned}
\]

with \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\) and \(y \in \mathbb{R}^k\), is called (internally) positive if and only if its state and output are nonnegative for every nonnegative input and every nonnegative initial state.

**Theorem: Positivity** [Luenberger, D. G., 1979]

A (cont.) linear system \((A, B, C, D)\) is positive if and only if \(A\) is a Metzler-matrix and \(B, C, D \succeq 0\).
"Fathers" of positive systems: Perron & Frobenius

Key result: Perron-Frobenius Theorem

(1849 - 1917) (1880 - 1975)
“ [...]the positivity property just defined, is always nothing but the immediate consequence of the nature of the phenomenon we are dealing with. A huge number of examples are just before our eyes.” [Farina, L., 2002]

- Network flows: traffic, transport, etc.
- Social science: population models
- Biology/Medicine: nitrade models, proteins, etc.
- Economy: stochastic models, markov jump systems, etc.
- Discretization of PDEs: heat equation
Example: Compartmental Network

\[ \dot{x}_i(t) = -k_{o,i}x_i(t) + \sum_{j \neq i}^n \left[ k_{ij}x_j(t) - k_{ji}x_i(t) \right] + \sum_{j=1}^m b_{ij}u_j(t) \]

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Positive and Monotone Systems
Publications: till 1999

Scopus: \( \sim 70 \) publications mentioning positive systems.

Important ones:

- Introduction to Dynamic Systems: Theory, Models & Applications. (Luenberger 1979, Wiley)
- Reachability, observability and realizability of continuous-time positive systems. (Ohta 1984, SIAM)
- Nonnegative Matrices in Dynamical Systems (Berman 1989, Wiley)
- Robust stability of positive differentiable linear systems (Son, Hinrichsen 1995, CDC)

However, the term 'positive system' was and is still not commonly used:

- Lyapunov Functions for Diagonally Dominant Systems. (Willems 1976, Automatica)
Publications: 2000 - today

Scopus: \(\sim 300\) publications mentioning positive systems.

Important ones:

- Positive Linear Systems (Farina 2000, Wiley)
- Stabilization of positive linear systems (De Leenheer 2001, Systems & Control Letters)
- Stability of continuous-time distributed consensus algorithms (Moreau 2008, CDC)

In Europe most of the research in Italy and Belgium, but also some in Lund:

- Distributed control of positive systems (Rantzer 2011, CDC)
- Some result on model reduction of positive systems (Aivar and myself 2012)

But much theory hidden in the application, i.a.

- Love dynamics: The case of linear couples (Rinaldi 1998, Applied Mathematics and Computations)
Still missing

Difficult to solve and still missing:

- Transfer of the SISO-theory to MIMO.
- Adequate realization algorithms.

So far some attempts, however under highly conservative restrictions - pretty messy theory!
Let \( \phi : X \subset V \to V \), where \( V \) is a real Banach space with an (partial) ordering \( x \geq y \) or a strongly ordering \( x \gg y \).

A dynamical system, with solution flow \( \phi \), is called **monotone** if \( \phi^t x \geq \phi^t y \) for \( t \geq 0 \) and \( x \geq y \) and **strongly monotone** if \( \phi^t x \gg \phi^t y \) for \( t > 0 \) and \( x \gg y \).

Proto-type: Cooperative system, which is the solution flow to a vector field \( F \) such that

\[
\frac{\partial F_i}{\partial x_j} \geq 0 \quad \text{for} \quad i \neq j.
\]

If \( x_i \) denotes the population of a species \( i \), then cooperative means, that an increase of \( x_i \) causes an increase in \( x_j \).
Early days: Hirsch, Smith & Smale

Key result: Convergence almost everywhere for strongly ordered systems (Hirsch 1981)

(Born 1930)  (Born 1933)
Publications: till 1999

Scopus: \(~ 230\) publications mentioning monotone and cooperative systems.

Among many convergence results:

- Cooperative systems of differential equations with concave nonlinearities (Smith 1985)
- Stability and convergence in strongly monotone dynamical systems (Hirsch 1988)
Scopus: ~ 1600 publications mentioning monotone and cooperative systems.

Important ones:
- Monotone control systems (Angeli, Sontag 2003, IEEE TAC)
- Monotone Dynamical Systems - Chapter 4, Handbook of Differential Equations (Hirsch, Smith 2005)

Nowadays most attention on: Communication, Coordination and Biology.

- IFAC2005: ~ 30 contributions (5 on positive systems)
- IFAC2008: ~ 30 contributions (3 on positive systems)
- IFAC2011: ~ 40 contributions (4 on positive systems)
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