Robust Control 2000  
Lecture 7

- RS and $H_\infty$ Optimization of Coprime Factors.
- $H_\infty$ Loop Shaping Procedure.
- Justification of $H_\infty$ Loop Shaping.

Robust Stabilization of Coprime Factors

Left coprime factor uncertainty model:

$$P_\Delta = (\tilde{M} + \tilde{\Delta} M)^{-1}(\tilde{N} + \tilde{\Delta} N).$$

By Small Gain Theorem:

RS for

$$\left( \begin{array}{c} K \\ I \end{array} \right) \left( I + PK \right)^{-1} \tilde{M}^{-1} \left( I + PK \right) \left( I + PK \right)^{-1} \tilde{M}^{-1} \left( I + PK \right) \leq \frac{1}{\epsilon},$$

This is $H_\infty$ optimization.

In the standard lower LFT form

$$\left( \begin{array}{c} K \\ I \end{array} \right) \left( I + PK \right)^{-1} \tilde{M}^{-1} = \mathcal{F}_I(G, K)$$

where

$$G = \left( \begin{array}{cc} 0 & I \\ \tilde{M}^{-1} & -P \end{array} \right).$$

State Space Formulas

Consider a state space representation of the strictly proper plant

$$P = \left( \begin{array}{cc} A & B \\ C & 0 \end{array} \right).$$

It is easy to verify that

$$[\tilde{N} \tilde{M}] = \left( \begin{array}{cc} A + LC & B \\ C & 0 \end{array} \right),$$

where $A + LC$ is stable, gives a left coprime factorization. Then

$$G = \left( \begin{array}{ccc} A & -L & B \\ 0 & I & 0 \\ C & I & 0 \end{array} \right).$$

Note: $D_{11} \neq 0$. Hence

$$\frac{1}{\epsilon_{opt}} = \gamma_{opt} > \|I\| = 1 \iff \epsilon_{opt} < 1.$$

Robust Stabilization of Coprime Factors

Apply $H_\infty$ optimization result to $G$ ([Zhou, Th. 14.7]). Two Hamiltonian matrices are

$$H_\infty = \left( \begin{array}{cc} A - \frac{1}{\gamma-1} LC & -\frac{1}{\gamma-1} LC^* \frac{1}{\gamma-1} LL^* - BB^* \\ -\frac{1}{\gamma-1} C^* C & -(A - \frac{1}{\gamma-1} LC)^* \end{array} \right),$$

$$J_\infty = \left( \begin{array}{cc} (A + LC)^* & -C^* C \\ 0 & -(A + LC) \end{array} \right).$$

Note that $Y_\infty = 0$. Thus the result becomes

**Theorem:** Let $D = 0$. Then there exists a stabilizing controller $K$ such that

$$\|\mathcal{F}_I(G, K)\|_\infty < \gamma$$

if and only if $\gamma > 1$, $H_\infty \in \text{dom}(\text{Ric})$ and $X_\infty = \text{Ric}(H_\infty) \geq 0$.

**Remark:** The result depends on the choice of $L$, i.e. choice of coprime factors.
Normalized Coprime Factors

Let choose $L$ such that $\tilde{M}$ and $\tilde{N}$ become the normalized left coprime factors.

Let $Y$ be the stabilizing solution to

$$AY + YA^* - YC^* CY + BB^* = 0.$$  

The matrix $A - YC^* C$ is stable, so we can put

$$L = -YC^*.$$  

**Lemma:** With the choice $L = -YC^*$ the left coprime factors become normalized.

**Proof:** Denote $\mathcal{A}(s) = (sI - A + YC^* C)^{-1}$ and calculate

$$\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = I - CAYC^* - CYA^* C + CA(BC + CYC)A^* C = I + CA(B'C + CYC - Y(A^*)^{-1} - A^{-1} Y)A^* C = I + CA(B'C + CYC + AY + YA')A^* C = I.$$  

Now calculate $\tilde{N}^*$ and put

$$A\text{ the matrix } H = \tilde{N}.$$  

Since $YQ$ is the square of the Hankel norm

$$\tilde{H}_{\infty}$$  

**Optimization of Normalized Coprime Factors**

**Theorem:** Let $D = 0$ and $L = -YC^*$ where $Y \geq 0$ is the stabilizing solution to

$$AY + YA^* - YC^* CY + BB^* = 0.$$  

Then $P = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization and

$$\inf_{K - \text{stab}} \left\| \begin{bmatrix} K & I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \frac{1}{\sqrt{1 - \lambda_{\text{max}}(YQ)}},$$

where

$$Q(A - YC^* C) + (A - YC^* C)^* Q + C^* C = 0.$$  

Moreover, a controller achieving $\gamma > \gamma_{opt}$ is

$$K(s) = \begin{bmatrix} A - BB^* X_{\infty} - YC^* C & -YC^* C \\ -B^* X_{\infty} & 0 \end{bmatrix},$$

$$X_{\infty} = \frac{\gamma^2}{\gamma^2 - 1} Q \begin{bmatrix} 1 & \gamma^2 \\ \gamma^2 & 1 \end{bmatrix}^{-1}.$$  

Some related $H_{\infty}$ problem

Since $\tilde{M}$, $\tilde{N}$ are the normalized lcf we have

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}^* (\tilde{M} \tilde{N}) = I.$$  

Therefore

$$\left\| \begin{bmatrix} K & I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\| = \left\| \begin{bmatrix} K & I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \left( \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \right) \right\| = \left\| \begin{bmatrix} K & I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|.$$  

Does not depend on factorization.

**Corollary:**

$$\inf_{K - \text{stab}} \left\| \begin{bmatrix} K & I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty = \frac{1}{\sqrt{1 - \lambda_{\text{max}}(YQ)}} = (1 - \|\tilde{N} \tilde{M}\|_{H}^2)^{-1/2}.$$

Note that $Y$ and $Q$ are controllability and observability Gramians for $[\tilde{N} \tilde{M}]$. Hence $\lambda_{\text{max}}(YQ)$ is the square of the Hankel norm of it.
**Right Coprime Factors**

What if we have normalized rcf $P = NM^{-1}$?

**Theorem:**

\[
\| \begin{pmatrix} I & K \\ P \\
\end{pmatrix} (I + PK)^{-1} \begin{pmatrix} I & P \\
\end{pmatrix} \| = \| \begin{pmatrix} I & P \\
K & I \\
\end{pmatrix} (I + KP)^{-1} \begin{pmatrix} I & K \\
P & I \\
\end{pmatrix} \|.
\]

**Corollary:** Let $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be the normalized rcf and lcf, respectively. Then

\[
\| \begin{pmatrix} K & I \\
I & P \\
\end{pmatrix} (I + PK)^{-1}\tilde{M}^{-1} \| = \| \tilde{M}^{-1}(I + KP)^{-1} \begin{pmatrix} I & K \\
P & I \\
\end{pmatrix} \|.
\]

**Conclusion:** It does not matter what kind of factorization we have. One can work with either.

---

**Stability Margin**

Introduce a quantity $b_{P,K}$

\[
b_{P,K} = \begin{cases} \| \begin{pmatrix} I & K \\ P \\
\end{pmatrix} (I + PK)^{-1} \begin{pmatrix} I & P \\
\end{pmatrix} \|_{\infty}^{-1} & \text{if } K \text{ stabilizes } P, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
b_{\text{opt}} = \sup_{K} b_{P,K}.
\]

Then $b_{P,K} = b_{K,P}$ and

\[
b_{\text{opt}} = \sqrt{1 - \lambda_{\text{max}}(YQ)} = \sqrt{1 - \|\tilde{N}\tilde{M}\|_{H}^2}.
\]

It holds $0 \leq b_{\text{opt}} \leq 1$. The larger $b_{\text{opt}}$ the more robustly stable the closed loop system.

This quantity is related to the classical gain and phase margins. Thus it can be considered as a general stability margin (Vinnicombe, 1993).

---

**Relation to Gain and Phase Margins**

**Theorem:** Let $P$ be a SISO plant and $K$ be a stabilizing controller. Then

- gain margin $\geq \frac{1 + b_{P,K}}{1 - b_{P,K}}$
- phase margin $\geq 2\arcsin(b_{P,K}).$

**Proof:** For SISO system at every $\omega$

\[
b_{P,K} = \frac{1}{\| \cdots \|_{\infty}} \leq \frac{1}{\| \cdots \|} = \| \begin{pmatrix} I & P \\
K & I \\
\end{pmatrix} \| = \| \begin{pmatrix} 1 & P \\
K & I \\
\end{pmatrix} \| = \frac{\| 1 + P(j\omega)K(j\omega) \|}{\| K \| \| 1 + P \|} = \frac{\| 1 + P(j\omega)K(j\omega) \|}{\sqrt{\| 1 + P(j\omega) \|^2 \| 1 + K(j\omega) \|^2}}.
\]

So at frequencies where $k := -PK \in R^+$ we have

\[
b_{P,K} \leq \frac{|1 - k|}{\sqrt{(1 + |P|^2)(1 + k^2/|P|^2)}} \leq \frac{|1 - k|}{\sqrt{\min\{(1 + |P|^2)(1 + k^2/|P|^2)\}}} = \frac{|1 - k|}{|1 + k|}
\]

from which the gain margin result follows.

Similarly at frequencies where $PK = -e^{\theta}$

\[
b_{P,K} \leq \frac{|1 - e^{\theta}|}{\sqrt{(1 + |P|^2)(1 + 1/|P|^2)}} \leq \frac{|1 - e^{\theta}|}{\sqrt{\min\{(1 + |P|^2)(1 + 1/|P|^2)\}}} = \frac{2|\sin(\theta/2)|}{2}
\]

which implies the phase margin result.
Loop-Shaping Design

Recall from Lecture 2 that a good performance controller design requires

- in low frequency region:
  \[ \sigma(PK) >> 1, \quad \sigma(KP) >> 1, \quad \sigma(K) >> 1 \]

- in high frequency region:
  \[ \sigma(PK) << 1, \quad \sigma(KP) << 1, \quad \sigma(K) \leq M \]

where \( M \) is not too large.

Conclusion: Good performance depends strongly on the open loop shape.

\( H_\infty \) loop shaping design procedure was suggested by Glover and McFarlane, 1990. The idea is to use pre- and postcompensators which give a desired open loop shape.

Loop Shaping Procedure

1) Choose \( W_1 \) and \( W_2 \) and absorb them into the nominal plant \( P \) to get the shaped plant \( P_s = W_2PW_1 \).
2) Calculate \( b_{opt}(P_s) = \sqrt{1 - \|N_s\|_{H_\infty}^2} \). If it is small then return to Step 1 and adjust weights.
3) Select \( \epsilon < b_{opt}(P_s) \) and design the \( H_\infty \) controller \( K_\infty \) such that

\[
\left\| \left[ \begin{array}{c}
I \\
K_\infty \\
\end{array} \right] \left( I + P_sK_\infty \right)^{-1}M_s^{-1} \right\|_{\infty} < \epsilon^{-1}.
\]

4) The final controller is \( K = W_1K_\infty W_2 \).

Remarks:

- In contrast to the classical loop shaping design we do not treat explicitly closed loop stability, phase and gain margins. Thus the procedure is simple.

- Observe that

\[
\left\| \left[ \begin{array}{c}
I \\
K_\infty \\
\end{array} \right] (I + P_sK_\infty)^{-1}M_s^{-1} \right\|_{\infty} = \left\| \left[ \begin{array}{c}
W_2 \\
W_1^{-1}K \\
\end{array} \right] (I + PK)^{-1} \left( W_2^{-1}PW_1 \right) \right\|_{\infty}.
\]

So it has an interpretation of the standard \( H_\infty \) optimization problem with weights.

- BUT!!! The open loop under investigation on Step 2 is \( W_2PW_1K_\infty \) and \( K_\infty W_2PW_1 \) whereas the actual open loop is given by \( W_1K_\infty W_2 \). This is not really what we have shaped!

Thus the method needs validation.

Justification of \( H_\infty \) Loop Shaping

We show that the degradation in the loop shape caused by \( K_\infty \) is limited.

Consider low-frequency region first.

\[
\sigma(PK) = \sigma(W_2^{-1}P_\infty W_2) \geq \frac{\sigma(P_\infty)\sigma(K_\infty)}{\kappa(W_2)},
\]

\[
\sigma(KP) = \sigma(W_1K_\infty P_\infty W_1^{-1}) \geq \frac{\sigma(P_\infty)\sigma(K_\infty)}{\kappa(W_1)}
\]

where \( \kappa \) denotes conditional number.

Thus small \( \sigma(K_\infty) \) might cause problem even if \( P_\infty \) is large. Can this happen?

Theorem: Any \( K_\infty \) such that \( b_{P_\infty,K_\infty} \geq 1/\gamma \) also satisfies

\[
\sigma(K_\infty) \geq \frac{\sigma(P_\infty) - \sqrt{\gamma^2 - 1}}{\sqrt{\gamma^2 - 1} - \kappa(P_\infty)} \quad \text{if} \quad \sigma(P_\infty) > \sqrt{\gamma^2 - 1}.
\]

Corollary: If \( \sigma(P_\infty) >> \sqrt{\gamma^2 - 1} \) then

\[
\sigma(K_\infty) \geq \frac{1}{\sqrt{\gamma^2 - 1}}.
\]
Consider now high frequency region.

\[ \bar{\sigma}(PK) = \bar{\sigma}(W_2^{-1}P_sK_{\infty}W_2) \leq \bar{\sigma}(P_s)\bar{\sigma}(K_{\infty})\kappa(W_2), \]

\[ \bar{\sigma}(KP) = \bar{\sigma}(W_1K_{\infty}P_sW_1^{-1}) \leq \bar{\sigma}(P_s)\bar{\sigma}(K_{\infty})\kappa(W_1). \]

Can \( \bar{\sigma}(K_{\infty}) \) be large if \( \bar{\sigma}(P_s) \) is small?

**Theorem:** Any \( K_{\infty} \) such that \( b_{P,K_{\infty}} \geq 1/\gamma \) also satisfies

\[ \bar{\sigma}(K_{\infty}) \leq \frac{\sqrt{\gamma^2-1} + \bar{\sigma}(P_s)}{1 - \sqrt{\gamma^2 - 1}\bar{\sigma}(P_s)} \quad \text{if} \quad \bar{\sigma}(P_s) < \frac{1}{\sqrt{\gamma^2 - 1}}. \]

**Corollary:** If \( \bar{\sigma}(P_s) << 1/\sqrt{\gamma^2 - 1} \) then

\[ \bar{\sigma}(K_{\infty}) \leq \sqrt{\gamma^2 - 1}. \]

One can get the idea of proof from SISO relation

\[ b_{P,K} \leq \frac{|1 + P_s(j\omega)K_{\infty}(j\omega)|}{\sqrt{1 + |P_s(j\omega)|^2} \sqrt{1 + |K_{\infty}(j\omega)|^2}}. \]

---

**What have we learned today?**

- \( H_{\infty} \) optimization of normalized coprime factors. Optimal value can be calculated via Hankel norm of the factors.
- Left or right coprime factors - does not matter.
- Stability margin \( b_{P,K} \). The larger the better. Relation to gain and phase margins.
- \( H_{\infty} \) loop shaping via pre- and postcompensations and optimization of \( b_{P,K} \).
- Relations \( PK, KP \leftrightarrow P_s, W \).

---

**Next lecture**

- Gap Metric and \( \nu \)-Gap Metric.
- Extended Loop Shaping.