The Least Measure of a Matrix Set with Applications

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Outline

- Background and Motivation
- The Least Measure
- Performance Estimate
- Summary
Absolute Stability for Lur'e Systems

- A Lur’e system is a linear plant with a sector-bounded nonlinear output feedback
- \([k_1,k_2]\)-absolute stability means global asymptotic stability with respect to the sector \([k_1,k_2]\)
- **Problem**: Determine the largest sector bound that guarantees absolute stability
- **Sub-problem**: Given the sector bound, verify whether the system is absolutely stable or not

\[
\dot{x}(t) = Ax(t) + b\varphi(y,t) \\
y(t) = cx(t) \\
k_1 y^2 \leq y\varphi(y,t) \leq k_2 y^2
\]
Classical Criteria

Frequency Domain

- Circle Criterion
- Popov Criterion

Nyquist plot stays within a circle
The Popov plot stays on the right of a line

Lyapunov Approach

- Quadratic
- Quadratic + Integral

\[ V(x) = x^T Px \]
\[ V(x) = x^T Px + \int_0^\sigma \phi(y)dy \]
Common Lyapunov Functions

The following statements are equivalent:

1. The Lur’e system is $[k_1,k_2]$-absolutely stable
2. There is a common Lyapunov function for $A+bk_1c$ and $A+bk_2c$

Molchanov & Pyatnitskiy (1986)

Each of the following function sets provides universal common Lyapunov functions for absolute stability:

- Convex and homogeneous of degree 2
- Polynomials
- Piecewise linear functions
- Piecewise quadratic functions
- Norms
Extensions

- **Switched Linear Systems**
  \[
  \dot{x}(t) \in \{ A_1 x(t), \ldots, A_m x(t) \}
  \]

- **Linear Differential Inclusions**
  \[
  \dot{x}(t) \in \text{co}\{ A_1 x(t), \ldots, A_m x(t) \}
  \]

The following statements are equivalent:

- The switched linear system is (asymptotically) stable
- The relaxed differential inclusion is (asymptotically) stable
- There is a common (strong) Lyapunov function for \( A_1, \ldots, A_m \)

Remarks

- Lyapunov approach has the full capacity in characterizing the stabilities
- Efficient searching of a Lyapunov function is hard (if not impossible)
- Tractable numerical verification is still unsolved
Largest Divergence Rate

- The largest divergence rate of a switched linear system is the largest possible rate of divergence with respect to all state trajectories.

\[ \rho = \sup \lim_{t \to \infty, \|x(0)\| = 1, \sigma} \ln \left| \frac{x(t)}{t} \right| / t \]

- For any switched linear system, the rate is always well-defined and bounded.
- The rate is connected to stabilities in an obvious manner.
Matrix Measure

For any vector norm $|\cdot|$, the induced matrix measure is

$$
\mu_{\| \cdot \|}(A) = \sup_{\tau \to 0+, |x|=1} \limsup_{\tau \to 0+} \frac{|x + \tau Ax| - |x|}{\tau}
$$

The measure is well defined, positively homogeneous, convex, and most importantly, satisfies

$$
|e^{At}| \leq e^{\mu_{\| \cdot \|}(A)t}, \quad \forall A, t \geq 0
$$
Common Measure of a Matrix Set

- Given a set of matrices $A = \{A_1, \cdots, A_m\}$, and a matrix measure $\mu$, the common measure of the matrix set is $\mu(A_1, \cdots, A_m) = \max \{\mu(A_1), \cdots, \mu(A_m)\}$

- The measure is well defined, positively homogeneous, convex, and satisfies
  $|e^{A_{i_1}t_1}e^{A_{i_2}t_2}\cdots e^{A_{i_k}t_k}| \leq e^{\mu_{\|A_1, \cdots, A_m\|}(t_1 + \cdots + t_k)}$

- The least measure is defined to be
  $\nu(A) = \inf_{\|\cdot\|} \max \{\mu_{\|\cdot\|}(A_1), \cdots, \mu_{\|\cdot\|}(A_m)\}$

- A measure is extreme if it is equal to the least measure
Main Result

- For any switched linear system, we have
  \[ \rho = \nu \]

- That is, the largest divergence rate is exactly the least possible common measure of the subsystem matrices set.

- For a singleton matrix, the property was established by Zahreddine (2003)
Outline of the Proof

- The least possible common measure is zero for marginally stable or marginally unstable systems.
- For marginally stable or marginally unstable systems, the largest divergence rate is zero.
- Other stability cases can be treated by a transition to the marginal case by a normalized transition, and use the linear properties of both the rate and the measure.
For any $n$-dimensional switched linear system, there is a polynomial $P$ with degree less than $n$ such that

$$|x(t)| \leq P(t)e^{\nu t} |x_0|$$

Moreover, $P$ can be chosen to be of degree zero iff the normalized switched system is marginally stable.

A marginally unstable system always admit an equivalent form of

$$\bar{A}_i \triangleq T^{-1} A_i T = \begin{bmatrix} \bar{A}_i^1 & \bar{A}_i^3 \\ 0 & \bar{A}_i^2 \end{bmatrix}$$

where both sub-modes are marginally stable as switched systems.
Stabilities in terms of the Least Measure

- The switched linear system is (exponentially) stable iff the least measure is negative.
- The switched linear system is marginally stable iff the least measure is zero and an extreme measure exists.
- The switched linear system is marginally unstable iff the least measure is zero and an extreme measure does not exist.
- The switched linear system is (exponentially) unstable iff the least measure is positive.
Computing the Least Measure

- Given $\epsilon > 0$, there exists a natural number $r \geq n$, a transformation matrix $X_{n \times r}$ of full row rank, and matrices $H_i \in \mathbb{R}^{r \times r}$, such that $A_i X = X H_i$

$$\mu_1(H_i) < V + \epsilon$$

Inspired by Blanchini (2000)

- Given $\epsilon > 0$, the least measure could be approximated with the given accuracy by means of a formula based on a sum-of-square with a sufficiently high degree

On-going study
Summary

- It was found that the least common matrices measure exactly characterizes the largest possible rate of divergence for the switched linear systems.
- This provides a more refined insight into the behavior of switched linear systems than stability.
- The result also applies to the linear convex differential inclusions, and Lur’e systems as well.

- Approximation of the least measure by means algebraic transformation and recursion.
Thank You

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