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Numerical Sensitivity of Linear Matrix Inequalities
Using Shift and Delta Operators
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Numerical Sensitivity of Linear Matrix Inequalities Using Shift and Delta Operators

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Outline

- Delta instead of shift operator for discrete-time dynamic systems
- Numerical sensitivity of Linear Matrix Inequalities
- Ill-conditioned LMI for shorter sampling periods
- Cancellation for shorter sampling periods
Delta operator

Continuous-time system

\[ \dot{x} = A_c x \]

Discrete-time system in the shift operator

\[ x(t_{k+1}) = A_q x(t_k) \]

\[ A_q = e^{hA_c} = \sum_{i=0}^{\infty} \frac{(hA_c)^i}{i!} = I + hA_c + O(h^2) \rightarrow I \quad \text{when} \quad h \rightarrow 0 \]

Discrete-time system in the delta operator

\[ \delta x(t_k) = \frac{x(t_{k+1}) - x(t_k)}{h} = \frac{A_q - I}{h} x(t_k) \triangleq A_\delta x(t_k) \]

\[ A_\delta = \frac{e^{hA_c} - I}{h} \rightarrow A_c \quad \text{when} \quad h \rightarrow 0 \]
Two shift operator models

Ordinary shift operator model

\[ G_q = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix} = \begin{bmatrix} I_n + hA_\delta & hB_\delta \\ C_\delta & D_\delta \end{bmatrix} \]

Signal scaled shift operator model. Input \( u_h = \sqrt{h}u \), output \( y_h = \sqrt{h}y \)

\[ G_q = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix} = \begin{bmatrix} I_n + hA_\delta & \sqrt{h}B_\delta \\ \sqrt{h}C_\delta & D_\delta \end{bmatrix} \]

The discrete \( \ell_2 \) norm then converges to the continuous \( L_2 \) norm

\[ \|y_h\|^2 = \sum_{k=0}^{\infty} y'_h(t_k)y_h(t_k) = \sum_{k=0}^{\infty} y'(t_k)y(t_k)h \]
Computing the $\mathcal{H}_\infty$ norm

For a stable discrete-time system $\mathcal{G}$, on shift operator form $G_q$, the $\mathcal{H}_\infty$ norm

$$\|\mathcal{G}\|_\infty = \max_\omega |G_q(e^{i\omega})|$$

With input $u$ and output $y$, the $\mathcal{H}_\infty$ norm is also given by the induced norm

$$\|\mathcal{G}\|_\infty = \sup_{\|u\| \neq 0} \frac{\|y\|}{\|u\|}$$

Then $\|\mathcal{G}\|_\infty < \gamma$, if and only if there exists a $P = P' > 0$ such that

$$P > A_q'PA_q + C_q'C_q - (A_q'PB_q + C_q'D_q)
B_q'PB_q + D_q'D_q - \gamma^2)^{-1}(A_q'PB_q + C_q'D_q)'$$

and

$$B_q'PB_q + D_q'D_q - \gamma^2 > 0$$
Computing the $\mathcal{H}_\infty$ norm using LMIs

Introducing the notation

\[
M_{q11}(P) = A_q' P A_q - P + C_q' C_q \\
M_{q12}(P) = A_q' P B_q + C_q' D_q \\
M_{q22}(P, \gamma) = B_q' P B_q + D_q' D_q - \gamma^2 I
\]

we obtain

\[
M_{q11}(P) - M_{q12}(P) M_{q22}^{-1}(P, \gamma) M_{q12}'(P) < 0 \\
M_{q22}(P, \gamma) > 0
\]

A Schur complement then gives the linear matrix inequality (LMI)

\[
M_q(P, \gamma) = \begin{bmatrix} M_{q11}(P) & M_{q12}(P) \\ M_{q12}(P) & M_{q22}(P, \gamma) \end{bmatrix} < 0
\]

Minimizing $\gamma$ satisfying this LMI is a semi-definite program, and the solution gives the $\mathcal{H}_\infty$-norm.
Corresponding LMI on delta operator form

\[
M_{\delta}(P, \gamma) = \begin{bmatrix}
M_{\delta_{11}}(P) & M_{\delta_{12}}(P) \\
M_{\delta_{12}}(P) & M_{\delta_{22}}(P, \gamma)
\end{bmatrix} < 0
\]

\[
M_{\delta_{11}}(P) = A_\delta'P + PA_\delta + hA_\delta'PA_\delta + C_\delta' C_\delta
\]

\[
M_{\delta_{12}}(P) = PB_\delta + hA_\delta'PB_\delta + C_\delta' D_\delta
\]

\[
M_{\delta_{22}}(P, \gamma) = hB_\delta'PB_\delta + D_\delta' D_\delta - \gamma^2 I
\]
Cancellation in shift operator LMI

For short sampling periods we have $A_q = I + O(h)$ and

$$M_{q11} = A'_q P A_q - P + C'_q C_q = (I + O(h))' P (I + O(h)) + hC'_\delta C_\delta - P$$

$$= P_h - P + O(h) = O(h)$$

where $P_h \approx P$.

Since $A_q = I + A_\Delta$ where $A_\Delta = hA_\delta$ this cancellation is avoided by replacing $M_{q11} = A'_q P A_q - P + C'_q C_q$ with

$$M_{\Delta 11} = A'_\Delta P + PA_\Delta + A'_\Delta PA_\Delta + C'_q C_q$$
Unbounded LMI solution when \( h \rightarrow 0 \)

\[
M_{q11}(P) = A_\delta' Ph + PhA_\delta + hA_\delta' PhA_\delta + C_\delta' C_\delta = M_{\delta 11}(Ph)
\]

\[
M_{q12}(P) = (I + hA_\delta) PhB_\delta + C_\delta' D_\delta = M_{\delta 12}(Ph)
\]

\[
M_{q22}(P, \gamma) = hB_\delta' PhB_\delta + D_\delta' D_\delta - \gamma^2 I = M_{\delta 22}(Ph, \gamma)
\]

Hence, we find that

\[
M_q(P, \gamma) = M_\delta(\bar{P}, \gamma)
\]

where \( \bar{P} = Ph \).

The solution \( P > 0 \) to the LMI \( M_q(P, \gamma) < 0 \) can alternatively be obtained as

\[
P = \frac{\bar{P}}{h}
\]
Semidefinite programming

LMIs are normally solved as convex optimization problems. Introduce the unknown variables $\xi = [\text{vec}(P)' \gamma]'$ which gives following semidefinite programming problem

$$\min \gamma$$

subject to $F(\xi) = \text{diag}(-M_q(P, \gamma), P) > 0$

where $F(\xi) \in \mathbb{R}^{m \times m}$ is symmetric and $m = (2n + n_u)$. Solved by an interior-point method with the barrier function

$$\phi(\xi) = -\log \det F(\xi)$$

The original criterion $\gamma$ is then replaced by the approximation

$$f(\xi) = \theta \gamma + \phi(\xi) = \theta \gamma - \log \det F(\xi)$$

where the approximation error is reduced when the parameter $\theta$ is increased.
Ill-conditioned problem

\[ M_q = \begin{bmatrix} \sqrt{hI_n} & 0 \\ 0 & I_{nu} \end{bmatrix} M_\delta \begin{bmatrix} \sqrt{hI_n} & 0 \\ 0 & I_{nu} \end{bmatrix} = T_h^{1/2} M_\delta T_h^{1/2} \Rightarrow \det M_q = h^n \det M_\delta \]

Since

\[ \det F(\xi) = \det \text{diag}(-M_q(P, \gamma), P) = (-1)^{n+nu} \det M_q(P, \gamma) \det P = (-1)^{n+nu} h^n \det M_\delta(P, \gamma) \det P \]

is close to zero independent of \( \gamma \) when \( h \) is small, we have an ill-conditioned problem for short sampling periods.

Solution: introduce the scaled LMI problem

\[ M_S(P, \gamma) = T_h^{-1/2} M_q(P, \gamma) T_h^{-1/2} = M_\delta(P, \gamma) < 0 \]

which gives the same optimal \( \gamma \) as the shift operator LMI, but without the singularity problem for small sampling periods.
Stored relative error using Hadamard (entry-wise) multiplication \( \circ \)

\[
A^\varepsilon = (\mathbf{1} + \varepsilon_A) \circ A
\]

Relative error due to subtraction

\[
(A - B)^\varepsilon = (\mathbf{1} + \varepsilon_s) \circ \left( (\mathbf{1} + \varepsilon_A) \circ A - (\mathbf{1} + \varepsilon_B) \circ B \right)
\]

Some manipulations then gives

\[
M_q^\varepsilon(P, \gamma) = T_h^\frac{1}{2} \left( M_\delta(P, \gamma) + \varepsilon \circ M_\delta(P, \gamma) + \frac{1}{h} \text{diag}(\varepsilon P \circ P, \mathbf{0}_{n_u \times n_u}) \right) T_h^\frac{1}{2}
\]
Error estimate

Hence we suggest the following relative error estimate for $\gamma_q$

$$e_\gamma = \frac{|\gamma - \gamma_0|}{\gamma_0} \approx \frac{\varepsilon_\gamma \Delta}{h} + \frac{\varepsilon_\gamma S}{h}$$

$$G_c(s) = \frac{12}{(s+1)(s^2+0.2s+1)(s^2+0.4s+4)}$$
Error estimate for the signal scaled model

![Graph showing relative error vs. sampling period h]
Conclusions

- The delta operator is excellent!

- For LMIs the system scaling part of the delta operator seems to be the most severe part from numerical point of view.