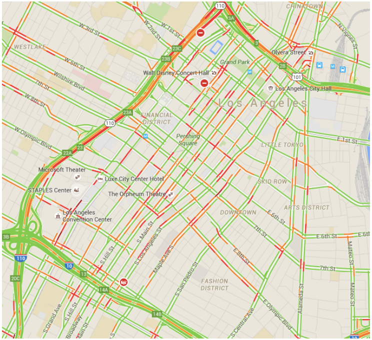
Entropy-Like Lyapunov Functions for the Stability Analysis of Adaptive Traffic Signal Controls

Gustav Nilsson, Lund University
Pouyan Hosseini, University of Southern California
Giacomo Como, Lund University
Ketan Savla, University of Southern California

54th IEEE Conference on Decision and Control
December 16, 2015

LUNDS UNIVERSITET

LUND LINNAEUS



Design traffic light feedback control that is

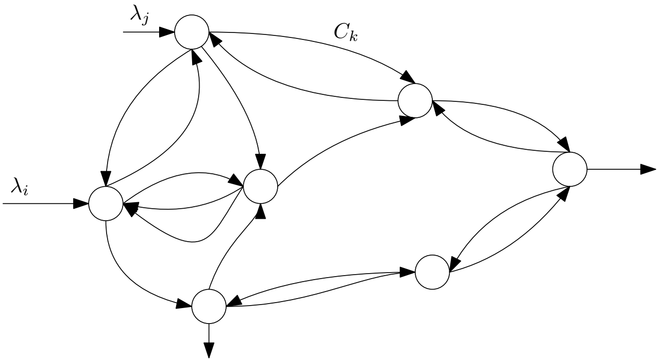
- Decentralized - Only depend on information nearby
- Scalable - Not depend on the network topology
- Throughput optimal - If possible, the controller should stabilize the network

Previous Work

- Max-pressure controller [Varaiya 2013, Tassiulas & Ephremides 1992]
  - The controllers have explicit information about the turning ratios
- Proportional controller [Savla et. al. 2013, 2014]
  - Acyclic networks
- Queueing networks [Massoulié 2007, Walton 2014]
  - Stochastic setting, computer networks with less physical constraints.

Outline

Model - Network



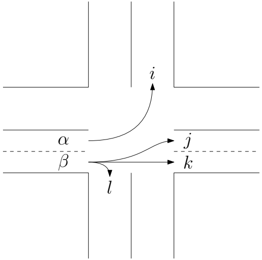
- Capacited multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$.
  - $\mathcal{V}$ - set of intersections
  - $\mathcal{E}$ - set of lanes
  - C - flow capacities
- External inflows λ.

Model - Routing matrix

- R is exogenous
- R_{ij} - fraction of flow from lane i to lane j
- $R_{ij} > 0 \Rightarrow j$ immediately downstream of i
- $\sum_{j \in \mathcal{E}} R_{ij} \leq 1$, where $1 - \sum_{j \in \mathcal{E}} R_{ij}$ is the fraction of the flow that will leave the network.
- The equilibrium flows can be computed by $a = (I - R^T)^{-1} \lambda$.

Example

- Lane α: devoted to left turns, $R_{\alpha i} = 1$.
- Lane β: both right turns and straight forward, $R_{\beta l} = 0.1$, $R_{\beta j} = 0.3, R_{\beta k} = 0.6$.

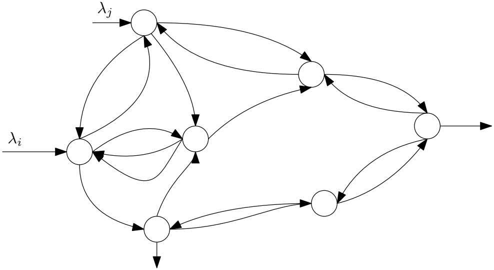


Model - Routing matrix

Assumption

(i) *All lanes can be reached by external inflow, i.e., for every $i \in \mathcal{E}$ there exists $h \in \mathcal{E}$ such that $\lambda_h > 0$ and $(R^l)_{hi} > 0$ for some $l \geq 0$.*

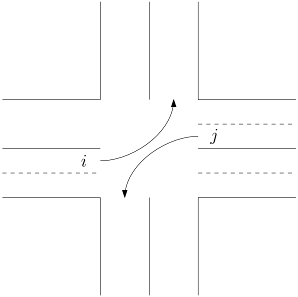
(ii) *It is possible to reach an exit from all lanes, i.e., for every $i \in \mathcal{E}$ there exists $k \in \mathcal{E}$ such that $\sum_{j \in \mathcal{E}} R_{kj} < 1$ and $(R^l)_{ik} > 0$ for some $l \geq 0$.*



Model - Phases

For each junction v, introduce a set of phases Ψ_v

- Set of binary vectors $p \in \{0,1\}^{\mathcal{E}_v}$.
- If it possible to activate lane i and j simultaneously, $p_i = p_j = 1$, $p_k = 0$ for all $k \in \mathcal{E}_v \setminus \{i,j\}$.
- Assumed to contain the zero phase, $0 \in \Psi_v$.



The controller's task is to determine the fraction each phase should be activated.

Model - Dynamics

- x_i density on lane i
- λ_i external inflow
- C_i the lanes capacity

$$\dot{x}_i = \lambda_i + \sum_{j \in \mathcal{E}} R_{ji} z_j(x) - z_i(x)$$

where $z_i(x) = C_i h_i(x)$ is the outflow from lane i and $1 \geq h_i(x) \geq 0$ determines the amount of green light lane i should receive:

$$h_i(x) = \sum_{p \in \Psi_v} \theta_p^{(v)}(x) p_i,$$

where $\sum_{p \in \Psi_v} \theta_p^{(v)}(x) = 1.$

Model - Maximizing green light policy

$\theta^{(v)}(x^{(v)})$ is determined by

$$\theta^{(v)}(x^{(v)}) \in \operatorname*{argmax}_{\theta \in \mathcal{S}_v} \sum_{i \in \mathcal{E}_v} x_i \log \left(\sum_{p \in \Psi_v} \theta_p p_i \right) + \kappa_v \log \theta_0 \,,$$

where $\mathcal{S}_v$ is the simplex of probability vectors over Ψ_v and $\kappa_v > 0$ is the weight on the zero phase.

Analysis Single Phase - Maximizing green light policy

- Every phase can prescribe green light to at most lane
- Set of phases

$$\Psi_v = \{p \in \{0,1\}^{\mathcal{E}_v} : \sum_{e \in \mathcal{E}_v} p_e \leq 1\}$$

- The maximizing green light policy

$$h_i^{(v)}(x^{(v)}) = \frac{x_i}{\sum_{j \in \mathcal{E}_v} x_j + \kappa_v}$$

Analysis Single Phase - Stability

Theorem

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a traffic network topology, R a routing matrix, λ an arrival vector satisfying the previous stated assumptions. Then the dynamical system, with maximizing green light policies, satisfying

$$\sum_{i \in \mathcal{E}_v} \frac{a_i}{C_i} < 1, \quad \forall v \in \mathcal{V}$$

admits a globally asymptotically stable equilibrium x^*.

Proof.

Idea: Use the Lyapunov function

$$V(x) = \sum_{i \in \mathcal{E}} x_i \log\left(\frac{z_i(x)}{a_i}\right) + \sum_{v \in \mathcal{V}} \kappa_v \log\left(\frac{h_0^{(v)}(x)}{h_0^{(v)}(x^*)}\right)$$

□

Multiphase Case - Analytical example

Local network, two incoming lanes

$$\dot{x}_1 = 1 - 2h_1(x)$$
$$\dot{x}_2 = 2 - 3h_2(x)$$

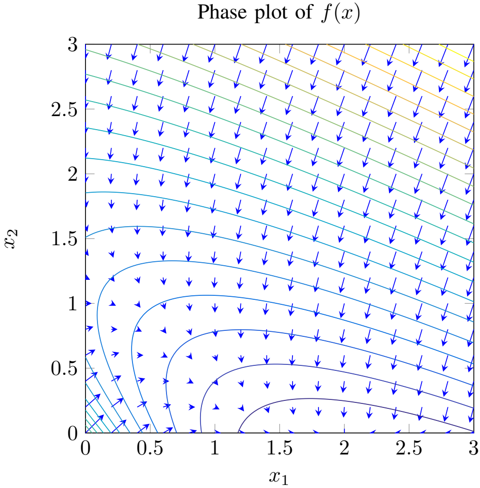
One common phase

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} \in \left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$$

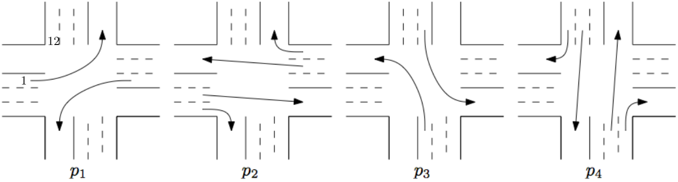
Explicit solution

$$h_1(x) = h_2(x) = \frac{x_1 + x_2}{x_1 + x_2 + \kappa}, \quad x > 0$$

Multiphase Case - Phase portrait

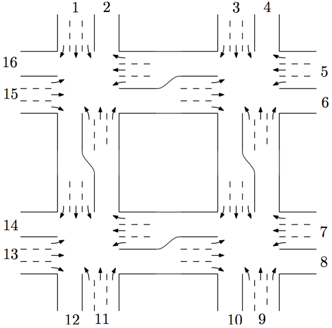


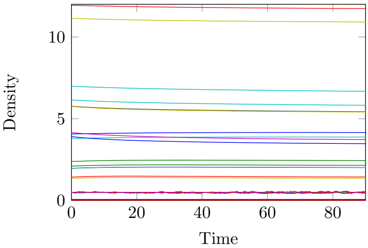
Multiphase Case - Network



$$\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T$$
$$\begin{bmatrix}0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\end{bmatrix}^T$$
$$\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}^T$$
$$\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1\end{bmatrix}^T$$

Multiphase Case - Network





Global stability is conjectured

Further Work

- Further theoretical investigation of the multiphase case
- Dynamic route choice behavior, i.e., R_{ij} depends on the state of the network
- Finite storage capacities
- Discrete time analysis
- Apply the controller to the Cell Transmission model/Supply-and-Demand model

Gustav Nilsson
Department of Automatic Control, Lund University
gustav.nilsson@control.lth.se