Scalable Control of Positive Systems

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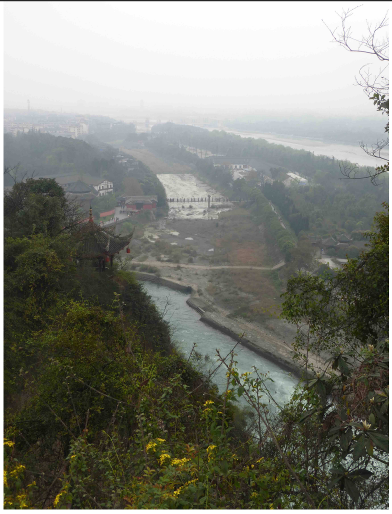


The Dujiangyan Irrigation System (250 B.C.)

Water — Still a Control Challenge!

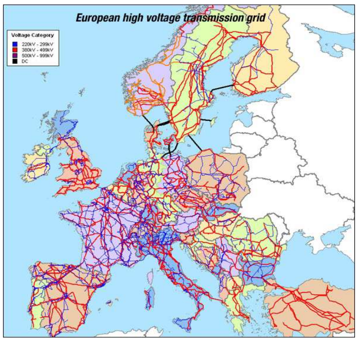
A scarce resource with different qualities for different needs:

- Drinking
- Washing
- Toilets
- Irrigation
- Industrial cooling
- ...

Many producers, many consumers in a complex network.

The Power Grid Needs Control

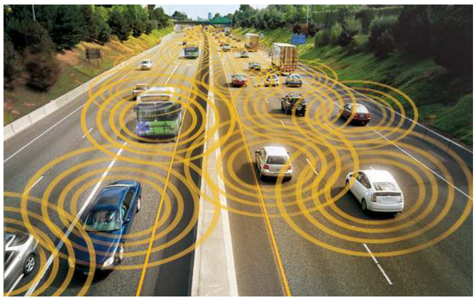
[Source: geospatial.blogs.com]



Control challenges: More producers. Variable capacity. Limited storage. Flexible components.

Traffic Networks Need Control

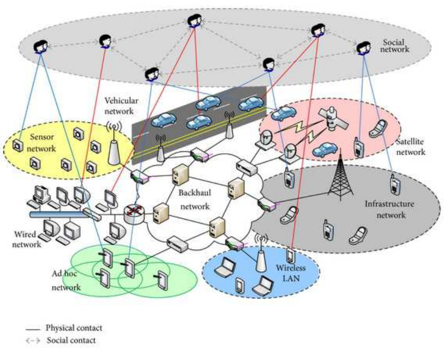
[Source: www.motorauthority.com]



Control challenges:
Throughput. Safety. Environmental footprint.

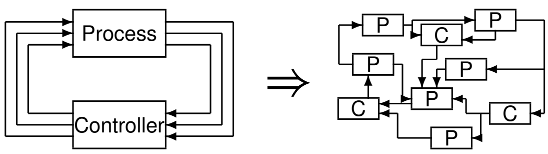
Communication Networks Need Control

[Source: www.hindawi.com]



Challenges: Service. Accessibility. Resource efficieny.

Towards a Scalable Control Theory



- Riccati equations use $O(n^3)$ flops, $O(n^2)$ memory
- Model Predictive Control requires even more
- **Today:** Exploiting monotone/positive systems

Outline

- **Linear Positive Systems**
  - Transportation networks
  - Vehicle formations
- Nonlinear Monotone Systems
  - Voltage stability
  - HIV/cancer treatment
- Frequency domain: Positively Dominated Systems
- Open problems and Conclusions

Positive systems

A linear system is called *positive* if the state and output remain nonnegative as long as the initial state and the inputs are nonnegative:

$$\frac{dx}{dt} = Ax + Bu \qquad\qquad y = Cx$$

Equivalently, A, B and C have nonnegative coefficients except possibly for the diagonal of A.

Examples:

- Probabilistic models.
- Economic systems.
- Chemical reactions.
- Ecological systems.

Positive Systems and Nonnegative Matrices

Classics:

Mathematics: Perron (1907) and Frobenius (1912)
Economics: Leontief (1936)

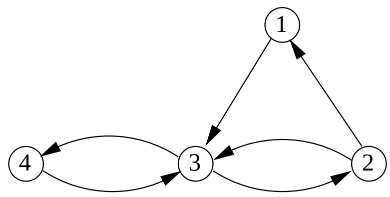
Books:

Nonnegative matrices: Berman and Plemmons (1979)
Large Scale Systems: Siljak (1978)
Positive Linear Systems: Farina and Rinaldi (2000)

Recent work on control of positive systems — Examples:

Biology inspired theory: Angeli and Sontag (2003)
Synthesis by linear programming: Rami and Tadeo (2007)
Switched systems: Liu (2009), Fornasini and Valcher (2010)
Distributed control: Tanaka and Langbort (2010)
Robust control: Briat (2013)

Example 1: A Transportation Network



$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0\\ 0 & -\ell_{12}-\ell_{32} & \ell_{23} & 0\\ \ell_{31} & \ell_{32} & -\ell_{23}-\ell_{43} & \ell_{34}\\ 0 & 0 & \ell_{43} & -4-\ell_{34}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix}$$

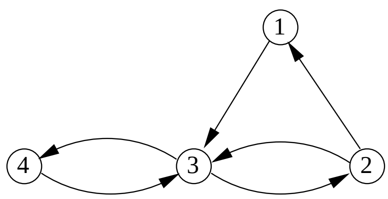
How do we select ℓ_{ij} to minimize some gain from w to x?

Transportation Network in Practice

- Irrigation systems
- Power systems
- Traffic flow dynamics
- Communication/computation networks
- Production planning and logistics

Example 2: A Vehicle Formation

Example 2: A Vehicle Formation



$$\begin{cases}\dot x_1 = -x_1 + \ell_{13}(x_3 - x_1) + w_1\\ \dot x_2 = \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) + w_2\\ \dot x_3 = \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) + w_3\\ \dot x_4 = -4x_4 + \ell_{43}(x_3 - x_4) + w_4\end{cases}$$

How do we select ℓ_{ij} to minimize some gain from w to x?

Stability of Positive systems

Suppose the matrix A has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There is a *diagonal* matrix $P \succ 0$ such that $A^T P + PA \prec 0$

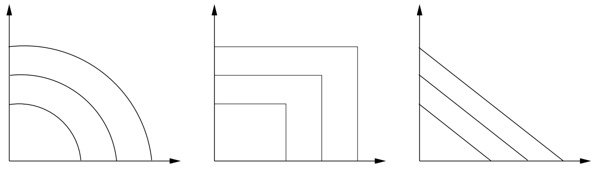
(iii) There exists a vector $\xi > 0$ such that $A\xi < 0$. (The vector inequalities are elementwise.)

(iv) There exits a vector $z > 0$ such that $A^T z < 0$.

Lyapunov Functions of Positive systems

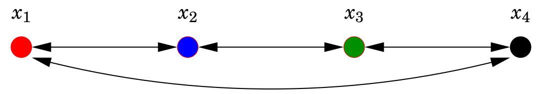
Solving the three alternative inequalities gives three different Lyapunov functions:

$A^TP + PA \prec 0$ $\qquad$ $A\xi < 0$ $\qquad$ $A^Tz < 0$



$V(x) = x^TPx$ $\qquad$ $V(x) = \max_k(x_k/\xi_k)$ $\qquad$ $V(x) = z^Tx$

A Scalable Stability Test



Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first node verifies the inequality of the first row.

The second node verifies the inequality of the second row.

...

Verification is scalable!

A Distributed Search for Stabilizing Gains

Suppose $\begin{bmatrix} a_{11}-\ell_1 & a_{12} & 0 & a_{14} \\ a_{21}+\ell_1 & a_{22}-\ell_2 & a_{23} & 0 \\ 0 & a_{32}+\ell_2 & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \geq 0$ for $\ell_1, \ell_2 \in [0, 1]$.

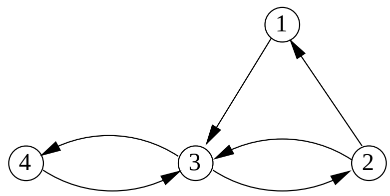
For stabilizing values of ℓ_1, ℓ_2, find $0 \leq \mu_k \leq \xi_k$ such that

$$\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and set $\ell_1 = \mu_1/\xi_1$ and $\ell_2 = \mu_2/\xi_2$. Every row gives a local test.

Distributed synthesis by linear programming (gradient search).

Optimal Control of Transportation Networks



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0 \\ 0 & -\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ \ell_{31} & \ell_{32} & -\ell_{23}-\ell_{43} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4-\ell_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + Bw$$

How do we select $\ell_{ij} \in [0, 1]$ to minimize some gain from w to Cx?

Performance of Positive systems

Suppose that $\mathbf{G}(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbb{R}^{n\times n}$ is Metzler, while $B \in \mathbb{R}_+^{n\times 1}$, $C \in \mathbb{R}_+^{1\times n}$ and $D \in \mathbb{R}_+$. Define $\|\mathbf{G}\|_\infty = \sup_\omega |G(i\omega)|$. Then the following are equivalent:

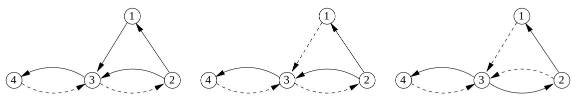
(i) The matrix A is Hurwitz and $\|\mathbf{G}\|_\infty < \gamma$.

(ii) The matrix $\begin{bmatrix} A & B \\ C & D-\gamma \end{bmatrix}$ is Hurwitz.

Example 1: Transportation Networks

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0 \\ 0 & -\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ \ell_{31} & \ell_{32} & -\ell_{23}-\ell_{43} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4-\ell_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} w \\ w \\ w \\ w \end{bmatrix}$$

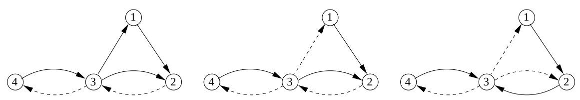
$$y = Cx$$



$C = \begin{bmatrix} 10 & 10 & 1 & 1 \end{bmatrix}$ $\qquad$ $C = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ $\qquad$ $C = \begin{bmatrix} 1 & 1 & 10 & 10 \end{bmatrix}$

Example 2: Vehicle Formations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1-\ell_{13} & 0 & \ell_{13} & 0 \\ \ell_{21} & -\ell_{21}-\ell_{23} & \ell_{23} & 0 \\ 0 & \ell_{32} & -\ell_{32}-\ell_{34} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4-\ell_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + Bw$$



$B = \begin{bmatrix} 10 \\ 10 \\ 1 \\ 1 \end{bmatrix}$ $\qquad$ $B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\qquad$ $B = \begin{bmatrix} 1 \\ 1 \\ 10 \\ 10 \end{bmatrix}$

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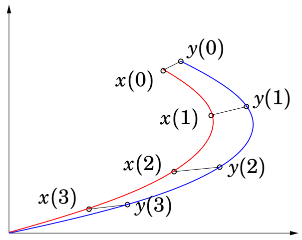
Nonlinear Monotone Systems

The system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = a$$

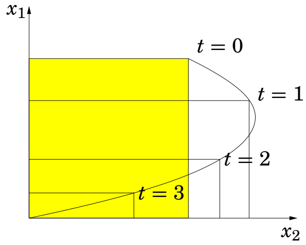
is a *monotone system* if its linearization is a positive system.



Max-separable Lyapunov Functions

Let $\dot{x} = f(x)$ be a globally asymptotically stable monotone system with invariant compact set $X \subset \mathbb{R}^n_+$. Then there exist strictly increasing functions $V_k : \mathbb{R}_+ \to \mathbb{R}_+$ for $k = 1, \dots, n$ with $\frac{d}{dt}V(x(t)) = -V(x(t))$ in X where $V(x)$ is equal to

$$V(x) = \max\{V_1(x_1), \dots, V_n(x_n)\}.$$



Voltage Stability

$$\begin{array}{c} \text{generator currents} \\ \text{load currents} \end{array} \quad \begin{bmatrix} -i^G(t) \\ i^L(t) \end{bmatrix} = \underbrace{\begin{bmatrix} Y^{GG} & Y^{GL} \\ Y^{LG} & Y^{LL} \end{bmatrix}}_{\text{network impedances}} \begin{bmatrix} u^G(t) \\ u^L(t) \end{bmatrix} \quad \begin{array}{c} \text{generator voltages} \\ \text{load voltages} \end{array}$$

$$\frac{di_k^L}{dt}(t) = \frac{p_k^*}{u_k^L(t)} - i_k^L(t) \qquad k = 1, \dots, n$$

Voltage stabilization is an important large-scale control problem.

Monotonicity be exploited for control synthesis!

Convex-Monotone Systems

The system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = a$$

is a *monotone system* if its linearization is a positive system. It is a *convex monotone system* if every row of f is also convex.

Theorem.

For a convex monotone system $\dot{x} = f(x, u)$, each component of the trajectory $\phi_t(a, u)$ is a convex function of (a, u).

Combination Therapy is a Control Problem

Evolutionary dynamics:

$$\dot{x} = \left(A - \sum_i u_i D^i\right) x$$

Each state x_k is the concentration of a mutant. (There can be hundreds!) Each input u_i is a drug dosage.

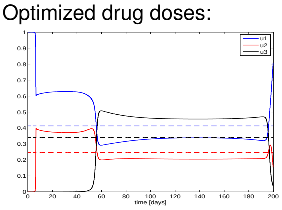
A describes the mutation dynamics without drugs, while $D^1, \dots, D^m \geq 0$ are diagonal matrices modeling drug effects.

Determine $u_1, \dots, u_m \geq 0$ with $u_1 + \dots + u_m \leq 1$ such that x decays as fast as possible!

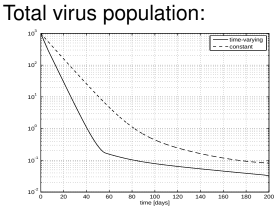
[Hernandez-Vargas, Colaneri and Blanchini, JRNC 2011]
[Jonsson, Rantzer, Murray, ACC 2014]

Example

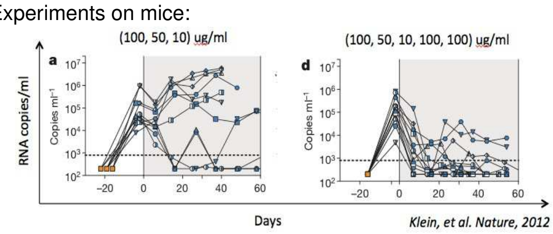
Optimized drug doses:



Total virus population:



Experiments on mice:



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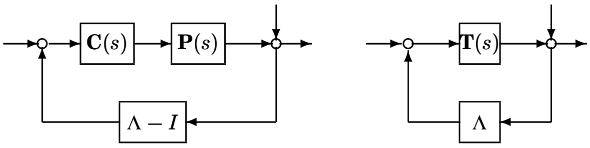
Platoon of Vehicles with Inertia

$$\begin{aligned}
(s^2 + 0.1s)x_1 &= -\mathbf{C}_1(s)x_1 + w_1 \\
(s^2 + 0.1s)x_2 &= \mathbf{C}_2(s)[\lambda_{21}(x_1 - x_2) + \lambda_{23}(x_3 - x_2)] + w_2 \\
(s^2 + s)x_3 &= \mathbf{C}_3(s)[\lambda_{32}(x_2 - x_3) + \lambda_{34}(x_4 - x_3)] + w_3 \\
(s^2 + s)x_4 &= -\mathbf{C}_4(s)x_4 + w_4
\end{aligned}$$

Negative feedback destroys positivity in second order models.

Is there a scalable approach to controller design?

General Approach



1. First design the local controllers to make $\mathbf{T} = \mathbf{PC}[I - \mathbf{PC}]^{-1}$ *positively dominated*:

$$|\mathbf{T}_k(i\omega)| \leq \mathbf{T}_k(0) \qquad \text{for all } k, \omega$$

2. Then use scalable methods to optimize the weights Λ.

Caveat: Not advisable for resonant systems.

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Open Problems

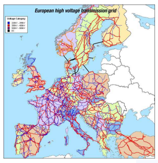
1. Sometimes **the same H_∞-performance can be attained** using distributed positivity based controllers as with Riccati based centralized controllers. Find out when!

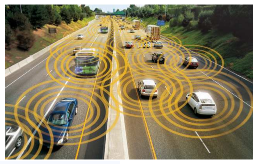
2. Overflow channels in waterways (and capacity limits in traffic networks) give rise to **piecewise linear monotone systems**. Find scalable methods to optimize their closed loop performance.

3. For **convex-monotone systems**, develop methods for design of controllers that give a monotone closed loop system with a separable Lyapunov function.

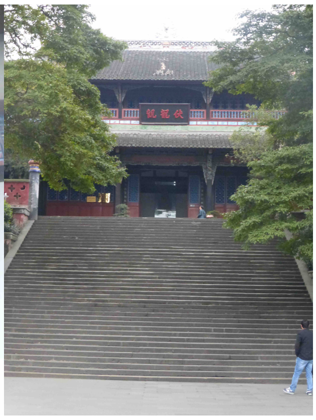
For Scalable Control — Use Positive Systems!

- Verification and synthesis scale well
- Distributed controllers by linear programming
- No need for global information
- Use local controllers to get positive dominance





Li Bing — Our Control Ancestor



Thanks!

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