LUND INSTITUTE OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - March 12, 2014, 8 am – 1 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

Preliminary grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/'"Collection of Formulae". Pocket memoryless calculator.

Note!

In many cases the sub-problems can be solved independently of each other.

Good Luck!

1. Consider the dynamical system

$$\begin{aligned}&\ddot{y} - 2\cos(\dot{y}) = -3y + 1\\&\dot{z} - 2\sin(z) = 0\end{aligned}$$

a. Write it in state-space form. (1 p)

b. Verify that $y = 1$, $z = 0$ is an equilibrium, and classify it. (1 p)

Solution

a. We can choose the following states $x_1 = y$, $x_2 = \dot{y}$, $x_3 = z$. Then, the dynamical system satisfies

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 2\cos(x_2) - 3x_1 + 1\\ \dot{x}_3 &= 2\sin(x_3)\end{aligned} \tag{1}$$

b. When $y = 1$, $\ddot{y} = \dot{y} = 0$, and $\dot{z} = z = 0$, one has that

$$\begin{aligned}&\ddot{y} - 2\cos(\dot{y}) = 0 - 2\cos(0) = -2 = -3y + 1\\&\dot{z} - 2\sin(z) = 0 - 2\sin(0) = 0\end{aligned},$$

so that the differential equations are satisfied. In order to classify the equilibrium, we linearize (1) around $x^* = (1, 0, 0)$. If $f(x) = (f_1(x), f_2(x), f_3(x))^T$, where

$$f_1(x) = x_2\,, \qquad f_2(x) = 2\cos(x_2) - 3x_1 + 1\,, \qquad f_3(x) = 2\sin(x_3)\,,$$

then its Jacobian matrix $\frac{\partial f}{\partial x}$ satisfies

$$\frac{\partial f}{\partial x}(x^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \frac{\partial f_1}{\partial x_2}(x^*) & \frac{\partial f_1}{\partial x_3}(x^*) \\ \frac{\partial f_2}{\partial x_1}(x^*) & \frac{\partial f_2}{\partial x_2}(x^*) & \frac{\partial f_2}{\partial x_3}(x^*) \\ \frac{\partial f_3}{\partial x_1}(x^*) & \frac{\partial f_3}{\partial x_2}(x^*) & \frac{\partial f_3}{\partial x_3}(x^*) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -2\sin(0) & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

whose eigenvalues are $3i$, $-3i$, and 2. Hence, the equilibrium is unstable since the Jacobian matrix has one eigenvalue with positive real part.

2. Consider the non-linear controlled system

$$\begin{aligned} \dot{x_1} &= -x_2 - x_1^3 + u_1 \\ \dot{x_2} &= x_1^5 + u_2 \,. \end{aligned} \tag{2}$$

a. Use exact feedback linearization to design a control law $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ making the origin $(0, 0)$ a globally asymptotically stable equilibrium for (2). (1 p)

b. Use the Lyapunov function candidate $V(x_1, x_2) = \frac{1}{2}(x_1 - x_1^*)^2 + \frac{1}{2}(x_2 - x_2^*)^2$, where you are free to choose the parameters x_1^* and x_2^*, in order to design a feedback control law $u_1(x_1, x_2)$ $u_2(x_1, x_2)$ making the point $(2, 1)$ a globally asymptotically stable equilibrium for the controlled dynamical system (2). (1 p)

Now, consider the uncontrolled system, i.e., the case where $u_1 = u_2 = 0$.

c. What conclusions does the linearisation method allow you to draw about the local stability properties of the equilibrium $(0, 0)$ for the uncontrolled system? (1 p)

d. Show that $(0, 0)$ is a globally asymptotically stable equilibrium for the uncontrolled system using the following Lyapunov function candidate

$$V(x_1, x_2) = \frac{x_1^6}{6} + \alpha \frac{x_2^2}{2}$$

where you need to choose a proper value of the parameter α. (2 p)

Solution

a. By cancelling all non-linearities with

$$u_1(x_1, x_2) = x_1^3 + x_2 - k_1 x_1, \quad k_1 > 0$$
$$u_2(x_1, x_2) = -x_1^5 - k_2 x_2, \quad k_2 > 0$$

we receive the linear asymptotically stable system

$$\dot{x}_1 = -k_1 x_1$$
$$\dot{x}_2 = -k_2 x_2$$

b. We use the Lyapunov-function $V(x_1, x_2) = \dfrac{(x_1 - 2)^2 + (x_2 - 1)^2}{2}$.

$$\begin{aligned}\dot{V}(x_1, x_2) &= (x_1 - 2)(-x_1^3 - x_2 + u_1) + (x_2 - 1)(x_1^5 + u_2) \\ &= -(x_2 - 2)^2 - (x_1 - 1)^2\end{aligned}$$

for $u_1 = x_1^3 + x_2 - (x_1 - 1)$ and $u_2 = -x_1^5 - (x_2 - 2)$. Then it holds, that

1. $\forall (x_1, x_2) \neq (1, 2) : V(x_1, x_2) > 0$
2. $V(1, 2) = 0$
3. $\forall (x_1, x_2) \neq (1, 2) : \dot{V}(x_1, x_2) < 0$
4. $V(x_1, x_2) \to \infty, \ \|(x_1, x_2)\|_2 \to \infty$

c. The Jacobian in $(0, 0)$ is given by

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

and has therefore all its eigenvalues in 0. Hence, we cannot make any conclusion about its stability.

d.

$$\begin{aligned}\dot{V} &= x_1^5 \dot{x}_1 + \alpha x_2 \dot{x}_2 \\ &= -x_1^8 + x_1^5(\alpha x_2 - x_2)\end{aligned}$$

Hence, for $\alpha = 1$ it follows that $\dot{V} = -x_1^8 \leq 0$ and therefore

1. $\forall (x, y) \neq (0, 0) : V(x_1, x_2) > 0$
2. $V(0, 0) = 0$
3. $\forall (x_1, x_2) : \dot{V}(x_1, x_2) \leq 0$
4. $V(x_1, x_2) \to \infty, \ \|(x_1, x_2)\|_2 \to \infty$

The set of all points, such that $\dot{V} = 0$ is obviously given by $M := \{(x_1, x_2) : x = 0\}$. For $(x_1, x_2) \in M \setminus \{(0, 0)\}$ it follows by the system dynamics, that $\dot{x}_1 \neq 0$. Hence, $\{(0, 0)\}$ is the largest invariant subset of M. By LaSalle's theorem and 1. - 4. we can conclude global asymptotic stability of $(0, 0)$.

3. The following dynamical system, known as the Lotka-Volterra model, is used to describe the dynamics of two populations of predators and preys:

$$\begin{aligned} \dot{x} &= x(\alpha - \beta y) \\ \dot{y} &= -y(\gamma - \delta x), \end{aligned} \tag{3}$$

where the two states $x(t)$ and $y(t)$ stand for the numbers of preys and predators, respectively, while α, β, γ, and δ are positive scalar parameters.

a. Determine and classify (stable/unstable node, focus, saddle, or center point) all equilibria of (3). (2 p)

b. Show that the function

$$V(x, y) = -\delta x + \gamma \log(x) - \beta y + \alpha \log(y)$$

is constant along trajectories of (3). (1 p)

c. Use the previous two points to determine the type of stability, or the instability, of the equilibria of the dynamical system (3). (1 p)

Solution

a. There are two equilibria: $(x_0^1, y_0^1) = (0, 0)$ and $(x_0^2, y_0^2) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$.

The Jacobian in (x_0^i, y_0^i) can be derived as

$$J(x_0^i, y_0^i) = \begin{pmatrix} \alpha - \beta y_0^i & -\beta x_0^i \\ \delta y_0^i & \delta x_0^i - \gamma \end{pmatrix}, \quad i = 1, 2$$

which gives

$$J(x_0^1, y_0^1) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \text{ and } J(x_0^2, y_0^2) = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ -\frac{\alpha\delta}{\beta} & 0 \end{pmatrix}$$

Hence, (x_0^1, y_0^1) is a saddle and (x_0^2, y_0^2) is a center.

b. We have

$$\begin{aligned} \dot{V} &= -\delta \dot{x} + \gamma \frac{\dot{x}}{x} - \beta \dot{y} + \alpha \frac{\dot{y}}{y} \\ &= -\delta x(\alpha - \beta y) + \gamma(\alpha - \beta y) + \beta y(\gamma - \delta x) - \alpha(\gamma - \delta x) \\ &= 0 \end{aligned}$$

c. Since $(0, 0)$ is a saddle point, it follows the point is unstable. The point $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ lies due to our assumptions strictly within the non-negative orthant. Hence, by the result in **b.** it follows, that $V(x, y)$ exists in neighbourhood of $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ and is constant along trajectories. Moreover, the level sets of V enclose $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$. Consequently, the system is locally stable, but not asymptotically stable.

To see, that V encloses $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ consider $W = -V$, which is also constant along trajectories. Then $\nabla W = \begin{pmatrix} \delta - \frac{\gamma}{x} \\ \beta - \frac{\alpha}{y} \end{pmatrix} = 0$ if and only if $(x, y) = (x_0^2, y_0^2)$. Furthermore, $\text{Hess}(x_0^2, y_0^2) = \begin{pmatrix} \frac{\delta^2}{\gamma} & 0 \\ 0 & \frac{\beta^2}{\alpha} \end{pmatrix} > 0$, which shows, that (x_0^2, y_0^2) is a minimum of W.

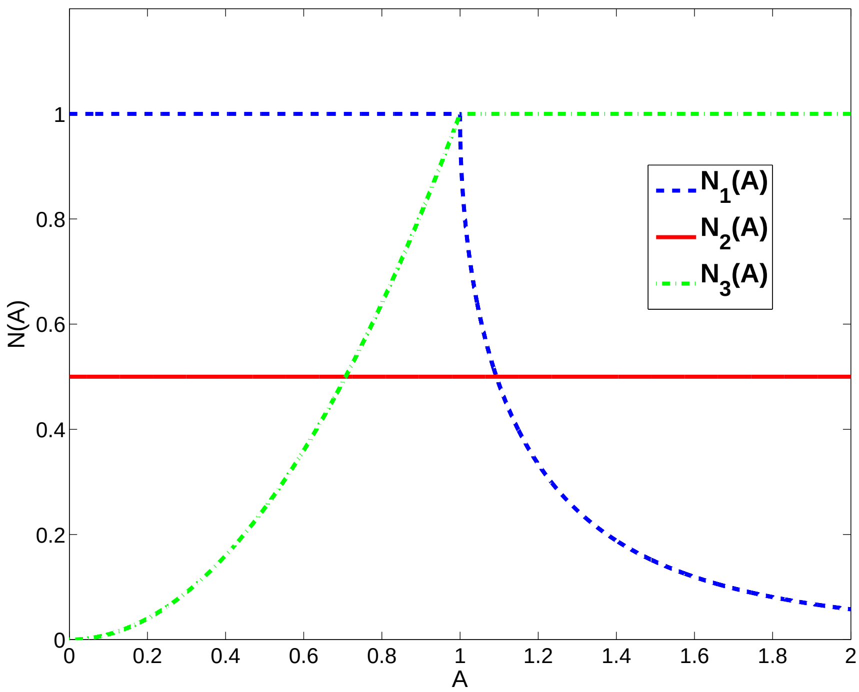


Figure 1 Candidate describing functions of Problem 4.

4.

a. The graphs of three different functions $N_1(A), N_2(A), N_3(A)$ are shown in Figure 1. Determine which graph corresponds to the static non-linearity whose graph is shown in Figure 2. Motivate your answer. (1 p)

b. Derive an explicit expression for the describing function of the static non-linearity whose graph is shown in Figure 2. (2 p)

c. Let the system in Figure 3 be given with $G(s) = \dfrac{10}{(s+1)(s^2+s+1)}$ and f as shown in Figure 2. The Nyquist plot of $G(s)$ is shown in Figure 4. Use the describing function method to predict the approximated frequency, amplitude and stability of all possible limit cycles for the output y. You are allowed to approximate the amplitude with the help of Figure 1. (2 p)

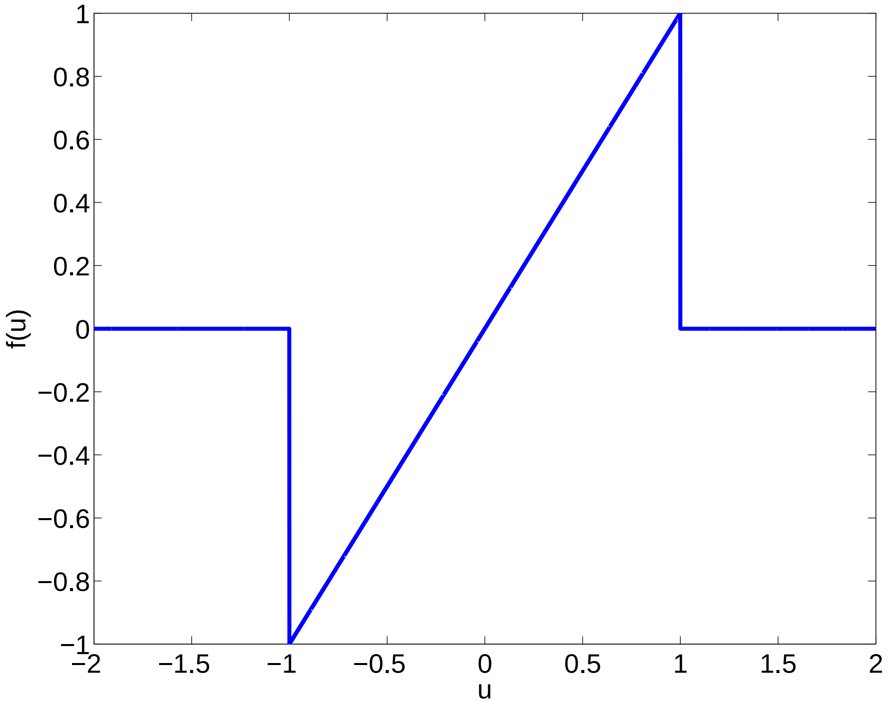


Figure 2 Non-linearity in Problem 4.

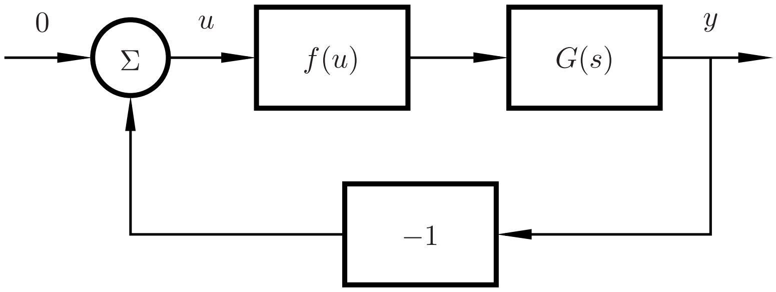


Figure 3 The system in Problem 4.c.

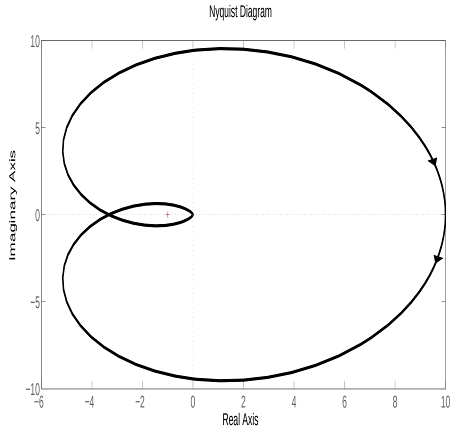


Figure 4 The Nyquist plot of $G(s) = \frac{10}{(s+1)(s+s^2+1)}$.

Solution

a. The right answer is describing function $N_1(A)$, because for small amplitudes A we get a liner behaviour, which implies, that $N(A)$ is constant. After that the describing function needs to decrease due to the zero-values of f.

b.

$$f(A\sin(\phi)) = \begin{cases} A\sin(\phi) & \phi \in [0, \phi_0) \cup (\pi - \phi_0, \pi + \phi_0) \cup (2\pi - \phi_0, 2\pi] \\ 0 & \text{else} \end{cases}$$

where $\phi_0 = \arcsin(\frac{1}{A})$. Since f is odd, it follows, that $a_1 = 0$. Let us now determine

$$b_1(A) = \frac{1}{\pi}\int_0^{2\pi} f(A\sin(\phi))\sin(\phi)d\phi$$

. If $A \leq 1$ then $f(A\sin(\phi)) = A\sin(\phi)$. In this case $b_1(A) = A$. Otherwise,

$$b_1(A) = \frac{4A}{\pi}\int_0^{\phi_0} \sin(\phi)^2 d\phi = \frac{2}{\pi}\left(A\phi_0 - \sqrt{1 - \frac{1}{A^2}}\right)$$

Hence, the describing function becomes

$$N(A) = \frac{b_1(A)}{A} = \begin{cases} 1 & A \leq 1 \\ \frac{2}{\pi}\left(\phi_0 - \sqrt{\frac{1}{A^2} - \frac{1}{A^4}}\right) & A > 1 \end{cases}$$

c. Since $G(s) = \dfrac{10}{(s+1)(s^2+s+1)}$ it follows, that

$$\begin{aligned} G(i\omega) &= \frac{10}{(i\omega+1)(-\omega^2+i\omega+1)} \\ &= 10\left(\frac{-2\omega^2+1}{|(i\omega+1)(-\omega^2+i\omega+1)|^2} + i\frac{\omega^3-2\omega}{|(i\omega+1)(-\omega^2+i\omega+1)|^2}\right) \end{aligned}$$

Hence, $G(i\omega)$ intersects the real axis if and only if $\omega = 0$ or $\omega = \sqrt{2}$. However, $G(0) > 0$ and therefore the frequency of a possible limit cycle can only be $\omega = \sqrt{2}$, which gives $G(i\omega) = 10\dfrac{-3}{|(i\sqrt{2}+1)(i\sqrt{2}-1)|^2} = -\dfrac{10}{3}$.

Therefore, it is necessary that $N(A) = \frac{3}{10}$ and we read of from Figure 1, that the amplitude should be $A \approx 1.2$. It is easy to see, that the limit cycle is stable.

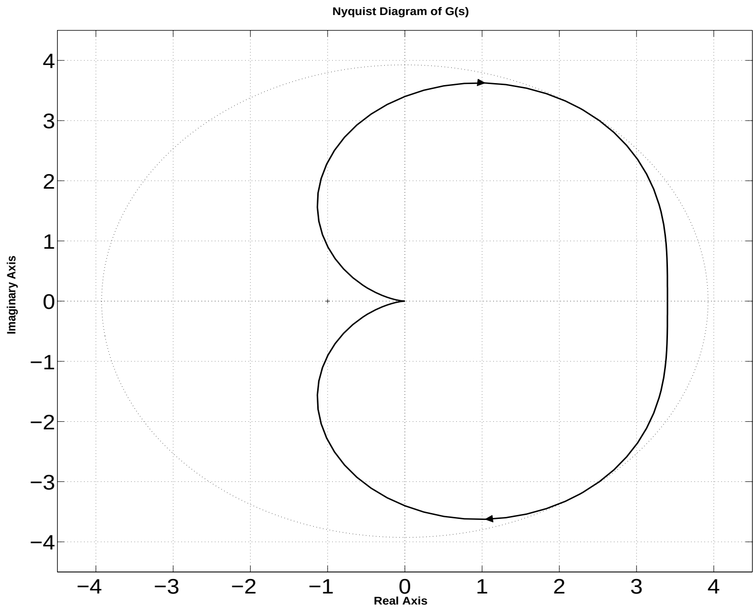
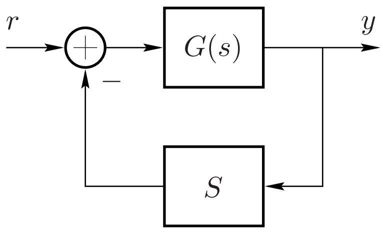


Figure 5 Nyquist plot for the linear system with transfer function $G(s) = \dfrac{3.4s}{1 + s + s^2}$.

5. The Nyquist plot of the linear system with transfer function $G(s) = \dfrac{3.4s}{1 + s + s^2}$ is shown in Figure 5. This system is fed back with a SISO system $u = S(y)$, according to the block diagram below:



For which of the following three choices of the feedback system S

1 S is a static nonlinearity $u(t) = f(y(t))$ where $f(y) = \frac{1}{2}\sin(y) + y$; (see Figure 6)

2 S is linear system with transfer function $G_S(s) = \dfrac{s}{4s + 4}$;

3 S is a static nonlinearity $u(t) = f(y(t))$ where f is odd and has a graph as in Figure 7;

can one determine stability of the closed-loop system using

a. the Small Gain Theorem? Please, specify the estimated gains. (2 p)

b. the Circle Criterion? Please, specify the estimated sector conditions for the non-linearities. (2 p)

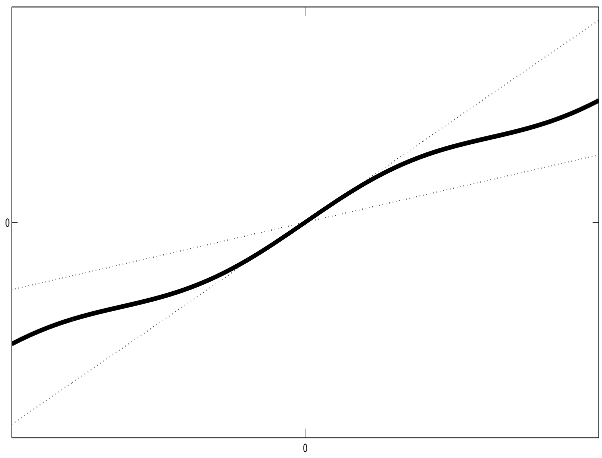


Figure 6 Graph of the static nonlinearity $f(y) = \frac{1}{2}\sin(y) + y$ in Problem 5.

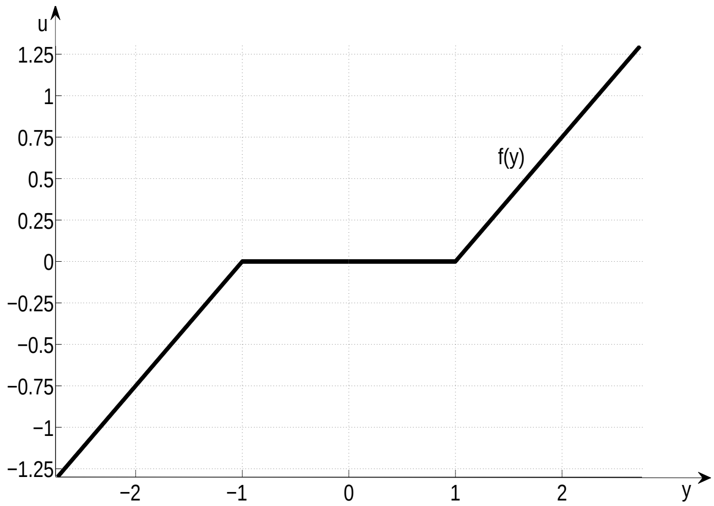


Figure 7 Graph of the static nonlinearity 3 in Problem 5.

Solution

a. The gain of a linear system with transfer function $G(s)$ is given by $\sup_{\omega>0} |G(i\omega)|$. In the case of the forward system in the problem, we easily get that $\gamma_G < 4$ by observing that, in Figure 5, there is a circle centered in 0 of radius $r < 4$ containing the Nyquist plot of G. Similarly, one easily gets a lower bound on γ_G, e.g., $\gamma_G \geq 3$. Then

$$3 \leq \gamma_G \leq r < 4\,.$$

Alternatively (and much less straightforwardly), one could have studied the real function $\frac{3.4\omega}{\sqrt{(1-\omega^2)^2+\omega^2}}$ in order to find its maximum value.

1 Recall that the gain of a static nonlinearity $u = f(y)$ is given by $\sup_{y\neq 0} \dfrac{|f(y)|}{|y|}$. One has that $|\sin(y)| \leq |y|$ so that

$$|f(y)| \leq |\frac{1}{2}\sin(y) + y| \leq |\frac{1}{2}\sin(y)| + |y| \leq 3/2|y|\,,$$

which implies that $\gamma_S \leq 3/2$. In fact,

$$\lim_{y\to 0} \frac{|f(y)|}{|y|} = \lim_{y\to 0} \frac{\frac{1}{2}\sin(y)}{y} + 1 = \frac{1}{2}\cos(1) + 1 = 3/2\,,$$

so that $\gamma_S = 3/2$. Since

$$\gamma_S\gamma_G \geq 9/2 > 1\,,$$

the Small Gain Theorem DOES NOT allow one to prove stability of the feedback interconnection.

2 As discussed above, the gain of the linear system S with transfer function $G_S(s)$ is given by

$$\gamma_S = \sup_{\omega>0} |G_S(i\omega)| = \sup_{\omega>0} \frac{\omega}{4\sqrt{1+\omega^2}}\,.$$

Now, since $\omega \leq \sqrt{1+\omega^2}$ for every ω, one gets that $\gamma_S \leq 1/4$. (In fact, one has that $\gamma_S = 1/4$ as can be checked by taking the limit as $\omega \to +\infty$. However, this is not needed here.) Then, one has that

$$\gamma_S\gamma_G \leq \frac{1}{4}\gamma_G < 1\,,$$

so that the Small Gain Theorem ALLOWS one to prove stability of the feedback interconnection.

3 As discussed in point **a.1** above, the gain of the static nonlinearity $u = f(y)$ is given by $\sup_{y\neq 0} \dfrac{|f(y)|}{|y|}$, which, for f as in Figure 7, equals the slope of the ramp that can be easily checked to be $0.75 = 3/4$. Then,

$$\gamma_S\gamma_G \geq 9/4 > 1\,,$$

so that the Small Gain Theorem DOES NOT allow one to prove stability of the feedback interconnection.

b. **1** Arguing as in point **a.1**, one gets that $|f(u)| \leq |\frac{1}{2}\sin(u)| + |u| \leq \frac{3}{2}|u|$. (As Figure 6 suggest, this is thigh since $f'(0) = 3/2$.) On the other hand, one can get a lower bound on $|f(u)|/|u|$, e.g., by noting that, since $|\sin u| \leq |u|$, one has $|f(u)| \geq |u| - \frac{1}{2}|\sin u| \geq \frac{1}{2}|u|$. This gives sector conditions

$$\gamma_1|y| \leq |f(y)| \leq \gamma_2|y|\,, \qquad \gamma_1 = \frac{1}{2}\,, \qquad \gamma_2 = \frac{3}{2}\,.$$

(One could have found a tighter value for γ_1, but this is enough to serve the purpose). Since the circle with diameter coinciding with the segment from $-1/\gamma_1 = -2$ to $-1/\gamma_2 = -2/3$ is outside the Nyquist plot of $G(s)$, one gets that, in this case, the Circle Criterium ALLOWS one to prove stability of the feedback interconnection.

2 In this case the system S is not static, so that the Circle Criterium DOES NOT ALLOW one to prove stability of the feedback interconnection.

3 Arguing as in point **a.3** gives sector conditions

$$\gamma_1|y| \leq |f(y)| \leq \gamma_2|y|\,, \qquad \gamma_1 = 0\,, \qquad \gamma_2 = 0.75\,.$$

(In this case γ_1 and γ_2 are tight, though this is irrelevant for our purpose.) Since the Nyquist plot lies coupletely on the righthand side of the line $\{z : \Re(z) = -1/\gamma_2 = -4/3\}$ in the complex plane, one gets that the Circle Criterium ALLOWS one to prove stability of the feedback interconnection.

6. Consider the following simplified model for an economy:

$$\begin{aligned} \dot{x}_1 &= -2x_2 - u \\ \dot{x}_2 &= -3x_2 + x_1 + u \,, \end{aligned}$$

where x_1 stands for the inflation rate, x_2 for the unemployment rate, and u for the interest rate. Assume that, given the current state $x_1(0) = 0.008$, $x_2(0) = 0.12$, the central bank will set the interest rate $u(t)$ during the time interval $0 \le t \le 1$ so as to minimize the cost

$$\int_0^1 ((1-\alpha)x_1(t) + \alpha x_2(t))\, \mathrm{d}t \,,$$

under the constraint that

$$0 \le u \le u_{\max} \,, \qquad 0 \le t \le 1 \,,$$

where $0 < \alpha < 1$ and $u_{\max} > 0$ are fixed parameters.

a. Write down the Hamiltonian for the optimal control problem above; (1 p)

b. Write down the co-state equations and their relative final time conditions; (1.5 p)

c. Solve the equations in point **b**. (1 p)

d. Now, assuming that an optimal control $u(t)$ exists for $0 \le t \le 1$, determine it for the following values of α (1.5 p)

1 $\alpha = 0.01$;

2 $\alpha = 0.2$;

3 $\alpha = 0.5$.

(Hint: you may find it convenient to look at the graphs of the functions $\lambda_1(t) - \lambda_2(t)$ drawn in Figure 8.)

Solution

The optimal control problem can be rewritten in the standard form as

$$\begin{aligned} &\min \int_0^{t_f} J\,(x(t), u(t))\, \mathrm{d}t + \Phi(x(t_f)) \\ &\dot{x} = f(x,u)\,, \qquad x_1(0) = x_1^0\,, \qquad x_2(0) = x_2^0 \\ &u(t) \in U\,, \end{aligned}$$

with

$$J(x,u) = (1-\alpha)x_1 + \alpha x_2\,, \qquad \Phi(x) = 0\,, \qquad t_f = 1\,, \qquad U = [0, u_{\max}]\,,$$

$$f_1(x,u) = -2x_2 - u\,, \qquad f_2(x,u) = -3x_2 + x_1 + u\,, \qquad x_1^0 = 0.008\,, \qquad x_2^0 = 0.12\,.$$

Since there is no final state constraint, and the the final time is given, we can use the first formulation of the maximum principle.

a. The Hamiltonian is given by

$$\begin{aligned} H(x_1, x_2, u, \lambda_1, \lambda_2) &= J(x_1, x_2, u) + \lambda_1 \tfrac{\partial}{\partial x_1} f_1(x_1, x_2, u) + \lambda_2 \tfrac{\partial}{\partial x_2} f_2(x_1, x_2, u) \\ &= (1-\alpha)x_1 + \alpha x_2 + \lambda_1(-2x_2 - u) + \lambda_2(-3x_2 + x_1 + u) \\ &= (\lambda_2 + 1 - \alpha)x_1 + (-2\lambda_1 - 3\lambda_2 + \alpha)x_2 + (\lambda_2 - \lambda_1)u \end{aligned}$$

b. The co-state equations are given by

$$\dot{\lambda_1} = -\frac{\partial}{\partial x_1} H(x_1, x_2, u, \lambda_1, \lambda_2) = -\lambda_2 + \alpha - 1\,,$$

$$\dot{\lambda_2} = -\frac{\partial}{\partial x_2} H(x_1, x_2, u, \lambda_1, \lambda_2) = 2\lambda_1 + 3\lambda_2 - \alpha\,,$$

with final time conditions

$$\lambda_1(1) = \frac{\partial}{\partial x_1}\Phi(x(1)) = 0\,, \qquad \lambda_2(1) = \frac{\partial}{\partial x_2}\Phi(x(1)) = 0\,.$$

c. By combining the co-state equations one gets that

$$\dot{\lambda}_1 + \dot{\lambda}_2 = 2(\lambda_1 + \lambda_2) - 1\,, \qquad \lambda_1(1) + \lambda_2(1) = 0\,,$$

so that

$$\lambda_1(t) + \lambda_2(t) = \frac{1}{2} - \frac{1}{2}e^{2(t-1)}\,.$$

Similarly,

$$2\dot{\lambda}_1 + \dot{\lambda}_2 = 2\lambda_1 + \lambda_2 + \alpha - 2\,, \qquad 2\lambda_1(1) + \lambda_2(1) = 0\,,$$

so that

$$2\lambda_1(t) + \lambda_2(t) = (2-\alpha) - (2-\alpha)e^{(t-1)}\,.$$

Hence, the co-state solutions are

$$\lambda_1(t) = -\alpha + \frac{3}{2} - (2-\alpha)e^{t-1} + \frac{1}{2}e^{2(t-1)}\,, \qquad 0 \le t \le 1\,,$$

$$\lambda_2(t) = -1 + \alpha + (2-\alpha)e^{t-1} - e^{2(t-1)}\,, \qquad 0 \le t \le 1\,.$$

ALT: Observe, that the co-state equations form a linear system $\dot{\lambda} = A\lambda + Bu$ with e.g. $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} \alpha - 1 \\ -\alpha \end{pmatrix}$ and $u = 1$. The solution to the linear system is given by

$$\lambda(t) = e^{At}\lambda(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

In order to solve this differential equation, one can use the collection of formula and notice, that the Laplace transform of e^{At} and $\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ is given by

$$(sI - A)^{-1} = \frac{1}{(s-1)(s-2)}\begin{pmatrix} s-3 & -1 \\ 2 & s \end{pmatrix}$$

and

$$(sI-A)^{-1}B\frac{1}{s} = \frac{1}{s(s-1)(s-2)}\begin{pmatrix} s(\alpha-1)-2\alpha+3 \\ -s\alpha+2\alpha-2 \end{pmatrix}$$

Again, with the help of the collection of formula we can inverse Laplace transform them and get

$$e^{At} = \begin{pmatrix} 2e^{t}-e^{2t} & e^{t}-e^{2t} \\ 2e^{2t}-2e^{t} & 2e^{2t}-e^{t} \end{pmatrix}$$

$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{pmatrix} \frac{1}{2}(e^t-1)(2\alpha+e^t-3) \\ -(e^t-1)(\alpha+e^t-1) \end{pmatrix}$$

Hence, for $t=1$, it follows, that

$$0 = \lambda(1) = \begin{pmatrix} 2e-e^2 & e-e^2 \\ 2e^2-2e & 2e^2-e \end{pmatrix}\lambda(0) + \begin{pmatrix} \frac{1}{2}(e-1)(2\alpha+e-3) \\ -(e-1)(\alpha+e-1) \end{pmatrix}$$

which gives, that

$$\lambda_1(0) = 0.5e^{-2} + e^{-1}(\alpha-2) - \alpha + 3/2$$

and

$$\lambda_2(0) = -e^{-2} + e^{-1}(2-\alpha) + \alpha - 1$$

Plugging it in, gives the same result as before.

d. Observe that the only term of the Hamiltonian that depends on u is $(\lambda_2-\lambda_1)u = -(\lambda_1-\lambda_2)u$. It follows from the previous point that

$$\lambda_1(t)-\lambda_2(t) = \frac{3}{2}e^{2(t-1)} - 2\alpha + \frac{5}{2} - 2(2-\alpha)e^{t-1}\,, \qquad 0 \le t \le 1\,.$$

Using the plots in Figure 8,[1] one gets that

1 For $\alpha = 0.01$, one has that $\lambda_1(t)-\lambda_2(t) < 0$ for all $0 \le t < 1$, so that $H(x_1, x_2, u, \lambda_1, \lambda_2)$ is minimized by $u(t) = 0$ for all $t \in [0,1]$.

2 For $\alpha = 0.2$, one has that there exists $t^* \in (0,1)$ such that $\lambda_1(t)-\lambda_2(t) > 0$ for all $0 \le t < t^*$, and $\lambda_2(t)-\lambda_1(t) < 0$ for all $t^* < t < 1$. Then, $H(x_1, x_2, u, \lambda_1, \lambda_2)$ is minimized by $u(t) = u_{\max}$ for $0 \le t < t^*$, and by $u(t) = 0$ for all $t^* < t \le 1$.

3 For $\alpha = 0.5$, one has that $\lambda_1(t)-\lambda_2(t) > 0$ for all $0 \le t < 1$, so that $H(x_1, x_2, u, \lambda_1, \lambda_2)$ is minimized by $u(t) = u_{\max}$ for all $t \in [0,1]$.

[1] Note: because of a misprint, in the exam the plots in Figure 8 where incorrectly referred to as the ones of $\lambda_2(t)-\lambda_1(t)$ instead of $\lambda_1(t)-\lambda_2(t)$. Correctly motivated solutions consistent with that interpretation (that correspond to switching the roles of 0 and $u_{\max}$ in the solution above) will of course be considered CORRECT.

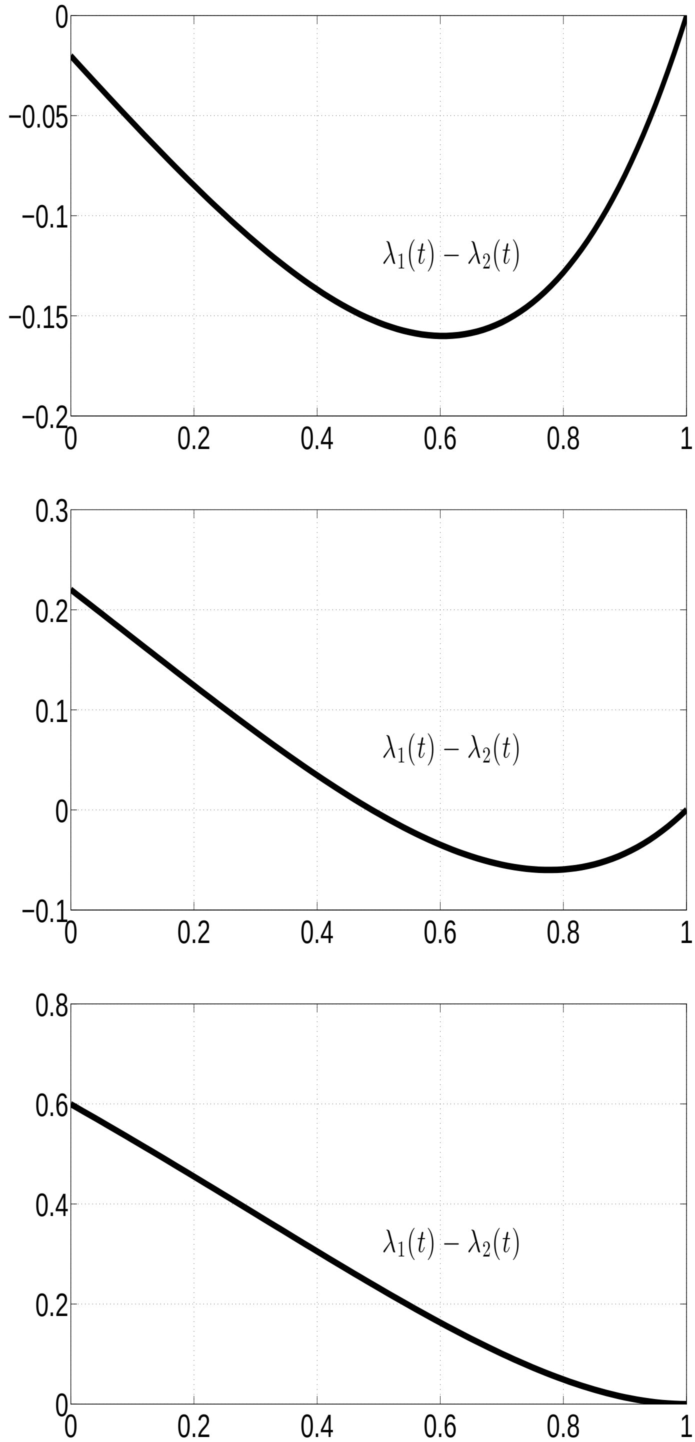


Figure 8 Graph of the function $\lambda_1(t) - \lambda_2(t)$ for $\alpha = 0.01$, $\alpha = 0.2$, and $\alpha = 0.5$.