LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - March 7, 2012, 8 am – 1 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

Preliminary grades:

3: 12 – 16 points

4: 16.5 – 20.5 points

5: 21 – 25 points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized “Formelsamling i reglerteknik”/”Collection of Formulae”. Pocket calculator.

Note!
In many cases the sub-problems can be solved independently of each other.

Good Luck!

Solutions to exam in Nonlinear Control March 7, 2012

1. Consider the system

$$\begin{aligned}\dot{x}_1 &= \arctan(x_1 + x_2)\\ \dot{x}_2 &= (x_1^2 + x_1 + x_2 - 1)x_1\end{aligned}$$

Determine all equilibrium points and their local stability properties. (2 p)

Solution

The equilibrium points are given as the solution to

$$\begin{aligned}0 &= \arctan(x_1 + x_2)\\ 0 &= (x_1^2 + x_1 + x_2 - 1)x_1\end{aligned}$$

Solving this gives the equilibrium points $x^0 = (0, 0)$ and $x^0 = (\pm 1, \mp 1)$. To determine their local stability properties, we linearize the system around each equilibrium point and check the eigenvalues of the A-matrix.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^0} = \left.\begin{pmatrix}\frac{1}{(x_1+x_2)^2+1} & \frac{1}{(x_1+x_2)^2+1}\\ 3x_1^2 + 2x_1 + x_2 - 1 & x_1\end{pmatrix}\right|_{x_0}$$

For each case we then have

- $x^0 = (0, 0) \Longrightarrow A = \begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix} \Longrightarrow \lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Unstable focus.
- $x^0 = (1, -1) \Longrightarrow A = \begin{pmatrix}1 & 1\\ 3 & 1\end{pmatrix} \Longrightarrow \lambda = 1 \pm \sqrt{3}$. Saddle point.
- $x^0 = (-1, 1) \Longrightarrow A = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \Longrightarrow \lambda = \pm\sqrt{2}$. Saddle point.

2.

a. Write the differential equation

$$\ddot{x} - 2(\dot{x})^2 + x = u - 1$$

in state-space form. (1 p)

b. Show that $x(t) = \cos(t)$, $u = \cos(2t)$ satisfies the differential equation and linearize the equation around this solution. (2 p)

Solution

a. We have a second order differential equation and thus we need two states. Introduce $x_1 = x$ and $x_2 = \dot{x}$.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + 2x_2^2 + u - 1\end{aligned} \tag{1}$$

b. $x(t) = \cos(t) \Rightarrow \dot{x}(t) = -\sin(t)$, $\ddot{x}(t) = -\cos(t)$. By just inserting these expressions and using that $\cos(2t) = 1 - 2\sin^2(t)$, we see that $x(t) = \cos(t), u = \cos(2t)$ is a solution. To linearize the system we can either originate from the original second order differential equation in the problem formulation, introduce the deviation from the nominal solution $\tilde{x} = x - \cos(t), \tilde{u} = u - \cos(2t)$, insert it in the equation and keep the terms linear in $\tilde{x}, \dot{\tilde{x}}, \ddot{\tilde{x}}$ and $\tilde{u}$.
An alternative way to solve the problem is to use the state-space form from subproblem **a**. By writing the state-space form like

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f(x_1, x_2, u)$$

we get

$$A = \frac{\partial f}{\partial x}_{|\{x_1=\cos(t), x_2=-\sin(t), u=\cos(2t)\}} = \begin{bmatrix} 0 & 1 \\ -1 & 4x_2 \end{bmatrix}_{|\{x_1=\cos(t), x_2=-\sin(t), u=\cos(2t)\}} = \begin{bmatrix} 0 & 1 \\ -1 & -4\sin(t) \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}_{|\{x_1=\cos(t), x_2=-\sin(t), u=\cos(2t)\}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so the linearization will be

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = A \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + B \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}$$

where $\tilde{x}_1 = x - \cos(t)$, $\tilde{x}_2 = x + \sin(t), \tilde{u} = u - \cos(2t)$

3.

a. Prove that the origin is globally asymptotically stable for the two systems below

$$\text{I:} \quad \begin{aligned} \dot{x}_1 &= -x_1^3 - x_1 x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned} \qquad \text{II:} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2^3 \end{aligned}$$

(2 p)

b. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - x_2 x_3 \\ \dot{x}_2 &= -x_2^5 + x_3 \tan x_1 \\ \dot{x}_3 &= x_1 x_2^2 + u \end{aligned}$$

Design a feedback controller $u(x)$ that renders the origin globally asymptotically stable. (2 p)

Solution

a. For each system we have:

- System 1: Choose for example $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. Its time derivative is given by

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1^3 - x_1x_2^2) - x_2^2 \\ &= -x_1^4 - x_1^2x_2^2 - x_2^2 < 0 \;\forall x \neq 0\end{aligned}$$

  Therefore the origin is guaranteed to be GAS.
- System 2: Use same Lyapunov function as in System I:

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_2^4 \leq 0$$

  Since $\dot{V} = 0 \;\forall x_1$ as long as $x_2 = 0$ we can not use the ordinary formulation of Lyapunovs stability theorem to prove that the origin is GAS. However, if we define the set $E = \{x|\dot{V}(x) = 0\} = \{x|x_2 = 0\}$ we see from the system dynamics that the largest invariant set in E is $\{0\}$. LaSalles invariant set theorem then tells us that the origin is GAS.

b. There are several ways of solving this problem, for example using Lyapunov theory with the Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^3)$. First, we notice that the requirements for being a Lyapunov candidate are fulfilled ($V(x) > 0,\ \forall x \neq 0,\ V(0) = 0,\ V(x) \to$ for $x \to \infty$). Furtheron, we require $\dot{V} \leq 0$.

$$\begin{aligned}\dot{V} &= x_1\dot{x_1} + x_2\dot{x_2} + x_3\dot{x_3} \\ &= x_1(-x_1^3 - x_2x_3) + x_2(-x_2^5 + x_3\tan x_1) + x_3(x_1x_2^2 + u) \\ &= -x_1^4 - x_2^6 + x_3(-x_1x_2 + x_2\tan x_1 + x_1x_2^2 + u)\end{aligned}$$

From this, it is obvious that if we choose

$$u(x) = x_1x_2 - x_2\tan x_1 - x_1x_2^2 - x_3$$

we get the $\dot{V} = -x_1^4 - x_2^6 - x_3^2 < 0 \;\forall x \neq 0$. Since all requirements for Lyapunov stability are fulfilled, we conclude that the origin is globally asymptotically stable with the chosen control law.

4. Figure 1 shows four odd static nonlinearities $f_1(u) - f_4(u)$ along with four describing functions $N_1(A) - N_4(A)$.

a. Match the nonlinearities with their respective describing function. A clear motivation is needed! (2 p)

b. Assume that each nonlinearity in Fig. 1 are feedback connected to a linear time-invariant system described by

$$G(s) = \frac{10}{(s+1)^3}$$

Investigate for each nonlinearity the possibility of a limit cycle. Also, state for each case the amplitude, frequency and stability of the limit cycle (if existing). Note that you can solve this subproblem even if you did not solve **a**. (2 p)

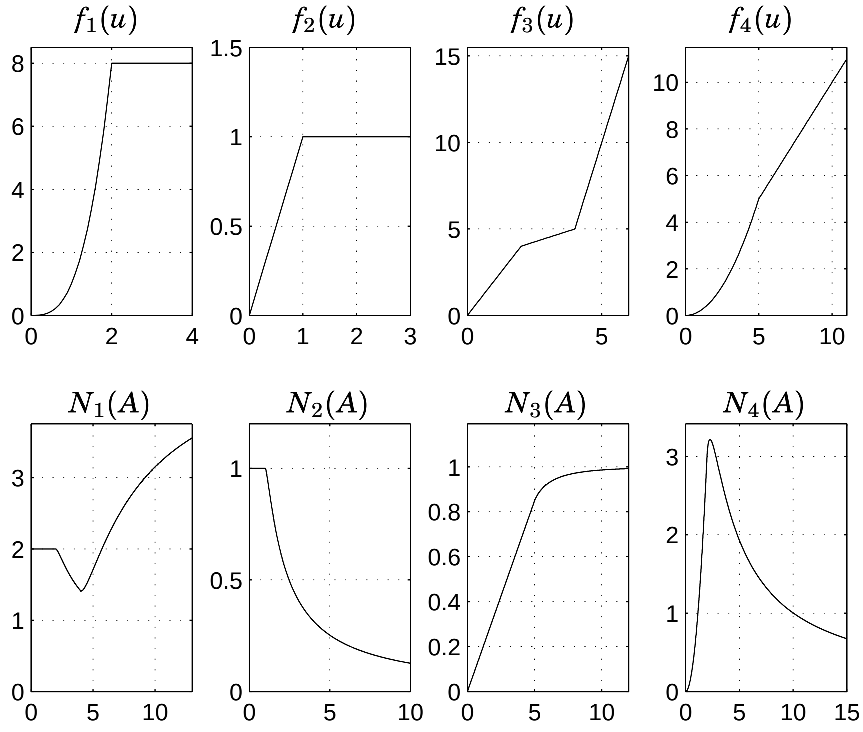

Figure 1 The nonlinearities and describing functions for problem 4.

Solution

a. If we adopt the view on describing function as effective gain we can say the following:

- $f_1(u)$: for $u < 2$ we have an increasing gain, starting from zero, due to the quadratic property of f_1, while for $u \geq 2$, the output is saturated, which means that the effective gain should decrease as $1/A$. This fits with $N_4(A)$.
- $f_2(u)$: this is a saturation. It has an effective gain of 1 for $A \leq 1$, and should then decrease as $1/A$. This is true for $N_2(A)$.
- $f_3(u)$: This nonlinearity starts with a linear part with slope 2, which then breaks down to 0.5 for $u = 2$. Later, at $u = 4$, it increases again to a slope >1. The corresponding describing function should then start at 2, then experience a decrease followed by an increase. This fits to N_1.
- $f_4(u)$: as for f_1, due to the quadratic behavior initially, we should expect the describing function to start at 0 and then increase rapidly. At $u = 5$, the nonlinearity starts to behave linearly with slope 1. Thus, for large A, we can expect the effective gain to approach 1. This holds for N_3.

b. Since all nonlinearities are odd and static, we can directly conclude that the frequency of any predicted limit cycle will be determined by the frequency for which the Nyquist curve $G(i\omega)$ cuts the negative real axis. Solving for this frequency and the systems gain for this frequency gives

$$\begin{aligned}\arg G(i\omega_0) &= -3\arctan\omega_0 = -\pi \Rightarrow \omega_0 = \sqrt{3} \\ |G(i\omega_0)| &= \frac{10}{4^{3/2}} = 1.25\end{aligned}$$

This means that whatever the nonlinearity is, any predicted limit cycle will have the frequency $\omega = \sqrt{3}$. The amplitude of the predicted limit cycle is given by the values of A that solve the equation

$$|G(i\omega_0)| = -\frac{1}{N(A)}$$

This implies that the describing function must contain the value $N(A) = 1/1.25 = 0.8$, which means that we immediately can rule out N_1. For N_2 we have $N_2(A) = 0.8$ for $A \approx 1.46$ and as the describing function is monotonically decreasing, $-1/N_2(A)$ will go from -1 to $-\infty$ for increasing A. This implies that there will be a stable limit cycle. For N_3, the intersection occurs for $A \approx 4.7$. As N_3 is monotonically increasing, $-1/N_3(A)$ will go from $-\infty$ to -1 for increasing A, thus implying an unstable limit cycle. An interesting case appears for $N_4(A)$, as $-N_4(A) = -0.8$ occurs for $A = 1.03$ and $A = 12.7$. As N_4 is increasing in the first case and decreasing in the second case, we will thus predict one unstable and one stable limit cycle with same frequency but different amplitudes.

5. Consider the second-order linear system

$$\begin{aligned}\dot{x}_1 &= -x_1 + 5x_2 \\ \dot{x}_2 &= -x_2 + u \\ y &= x_1\end{aligned}$$

The corresponding Nyquist plot is shown in Fig. 2. The system is connected in feedback with a non-linear function $f(y)$ according to $u = -f(y)$ where

$$f(y) = \begin{cases} \log(1+y) & y > 0 \\ 0 & y \le 0 \end{cases}$$

a. Does stability of the closed loop system follow from the circle criterion? (1.5 p)

b. Does stability of the closed loop system follow from the Small Gain Theorem? (1.5 p)

Solution

a. $f(y)$ can be bounded to the sector $[0, 1]$. The circle criterion thus guarantees stabililty since the Nyquist plot stays to the right of the vertical line that passes through the point -1.

b. $f(y)$ has gain less than or equal to one for all inputs, thus the Small Gain theorem would give stability if the linear system had gain ≤ 1 aswell. This is however not the case as can be seen in Fig. 2.

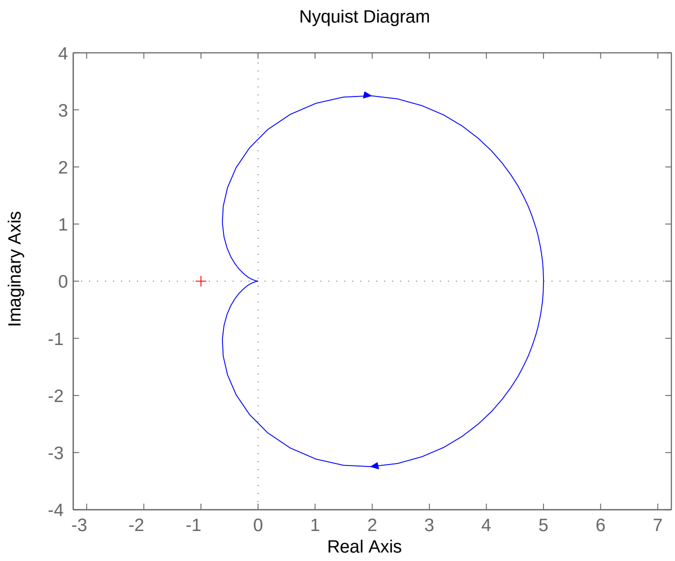

Figure 2 Nyquist plot for system in problem 5.

6. Consider a system with a 3-dimensional state $x = (x_1, x_2, x_3)$. Assume that the point $x = 0$ is an equilibrium. For which of the following five Lyapunov function candidates V does verification of the inequality $\dot{V}(x) < -V(x)$ in a neighborhood of $x = 0$ imply asymptotic stability of the equilibrium?

1. $V(x) = x_1^2 + x_2^2 + (x_3 - x_2)^2$
2. $V(x) = (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2$
3. $V(x) = (x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2$
4. $V(x) = x_1 + x_2 + x_3$
5. $V(x) = -x_1^2 - x_2^2 - x_3^2$

(2 p)

Solution

1. Valid, $V(0) = 0, V(x) \geq 0, V$ unbounded for large x
2. Invalid, $V(0) \neq 0$
3. Valid, $V(0) = 0, V(x) \geq 0, V$ unbounded for large x
4. Invalid, $V(x) < 0$ for some x (take for example $(-1, -1, -1)$)
5. Invalid, $V(x) \leq 0$ for all x

7. Consider the sliding mode controlled system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 7 & \eta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(x_1, x_2),$$

where η is a scalar parameter and

$$u(x_1, x_2) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \text{ if } x_1 < 0, \qquad u(x_1, x_2) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \text{ if } x_1 > 0.$$

a. Determine the sliding set. (1.5 p)

b. Complete the definition of $u(x_1, x_2)$ by computing the sliding dynamics. (1.5 p)

c. For which values of the parameter η is $x^* = [0,0]^T$ a stable equilibrium for the sliding dynamics? (1 p)

Solution

Let us rewrite the system as

$$\dot{x} = \begin{cases} f^+(x) & \text{if} \quad x_1 > 0 \\ f^-(x) & \text{if} \quad x_1 < 0 \end{cases},$$

with

$$f^+(x) = \begin{bmatrix} 0 & 2 \\ 7 & \eta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \qquad f^-(x) := \begin{bmatrix} 0 & 2 \\ 7 & \eta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

a. The sliding surface is $\{x_1 = 0\}$, with normal vector $[1,0]^T$. In order to determine the sliding set, we need to find the subset of the sliding surface where

$$[1,0]f^+(x) < 0, \qquad [1,0]f^-(x) > 0.$$

The above gives $2x_2 - 1 < 0$, and $2x_2 + 1 > 0$, i.e., $-1/2 < x_2 < 1/2$. Hence, the sliding set is $\{(0, x_2) : |x_2| < 1/2\}$.

b. The sliding dynamics are given by the convex combination

$$\dot{x} = \alpha f^+(x) + (1-\alpha) f^-(x),$$

where $\alpha = \alpha(x)$ is determined by the condition

$$[1,0]\left(\alpha f^+(x) + (1-\alpha) f^-(x)\right) = 0.$$

The above gives

$$2x_2 - \alpha + (1-\alpha) = 0,$$

that is $\alpha = x_2 + 1/2$. Substituting back, one finds that the sliding dynamics is given by

$$\dot{x} = \alpha f^+(x) + (1-\alpha) f^-(x) = \begin{bmatrix} 0 & 0 \\ 7 & \eta + 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c. On the sliding set the dynamics are given by $\dot{x}_2 = (\eta + 6)x_2$. The origin is a locally stable equilibrium if and only if $\eta \leq -6$, and is an asymptotically stable equilibrium if and only if $\eta < -6$.

8. The velocity $x(t)$ of a moving particle affected by viscous friction and a driving force $u(t)$ is described by the relation

$$\dot{x}(t) = \alpha x(t) + \beta u(t)\ , \ x(0) = x_0$$

where α and β are parameters. In a certain application, the following cost function is to be minimized

$$J = \int_0^{t_f} u^2(t)\ \mathrm{d}t + \gamma x^2(t_f)$$

where t_f is given and $\gamma > 0$ is a parameter determining the trade-off between the cost of utilization of control signal and the terminal cost.

a. Consider the special case when the particle is initially at rest, *i.e.*, $x(0) = 0$. What is the optimal control signal $u^*(t)$ in this case? (0.5 p)

b. Assuming that $x(0) \neq 0$, determine a relationship between the optimal control signal $u^*(t)$ and $x^*(t_f)$. (1.5 p)

c. Using the optimal control signal $u^*(t)$ calculated in part **b**, determine the optimal state trajectory $x^*(t)$, and in particular $x^*(t_f)$. What happens if we let $\gamma \to \infty$? What is the intuition behind this? (1 p)

Solution

a. We note that the minimum of the cost function J is 0. This value is achieved for $u(t) \equiv 0$, since this choice of control signal gives $x(t_f) = x(0) = 0$.

b. Introduce

$$L(x(t), u(t)) = u^2(t) \qquad \text{and} \qquad \phi(x(t_f)) = \gamma x^2(t)$$

and the Hamiltonian

$$H(x(t), u(t), \lambda(t)) = L(x(t), u(t)) + \lambda(t)(\alpha x(t) + \beta u(t))$$

where $\lambda(t)$ is the adjoint variable. From the Maximum Principle the following relations are obtained

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -\alpha\lambda(t) \qquad \text{and} \qquad \lambda(t_f) = \frac{\partial \phi(x^*(t_f))}{\partial x} = 2\gamma x^*(t_f)$$

Solving this first order differential equation gives the expression

$$\lambda(t) = 2\gamma x^*(t_f) e^{\alpha(t_f - t)}$$

Further, it is known that the optimal control signal $u^*(t)$ satisfies the relation

$$\frac{\partial H}{\partial u} = 2u^*(t) + \beta\lambda(t) = 0$$

from which it follows that

$$u^*(t) = -\frac{1}{2}\beta\lambda(t) = -\beta\gamma x^*(t_f) e^{\alpha(t_f - t)}$$

c. Inserting the optimal control signal $u^*(t)$ in the system dynamics gives

$$\dot{x}(t) = \alpha x(t) - \beta^2 \gamma x(t_f) e^{\alpha(t_f - t)}$$

Multiplication of both sides in the equation above with the integrating factor $e^{-\alpha t}$ gives the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\alpha t} x(t)\right) = -\beta^2 \gamma x(t_f) e^{\alpha(t_f - 2t)}$$

from which it follows that

$$e^{-\alpha t} x(t) = \frac{\beta^2 \gamma x^*(t_f)}{2\alpha} e^{\alpha(t_f - 2t)} + D$$

Consequently, the optimal state trajectory can be written

$$x^*(t) = \frac{\beta^2 \gamma x^*(t_f)}{2\alpha} e^{\alpha(t_f - t)} + D e^{\alpha t}$$

In order to determine the constant D, the initial condition $x^*(0) = x_0$ is utilized. We get

$$D = x_0 - \frac{\beta^2 \gamma x^*(t_f)}{2\alpha} e^{\alpha t_f}$$

Hence,

$$x^*(t) = x_0 e^{\alpha t} + \frac{\beta^2 \gamma}{2\alpha} x^*(t_f) \left(e^{\alpha(t_f - t)} - e^{\alpha(t_f + t)}\right)$$

Finally, $x^*(t_f)$ is determined by setting $t = t_f$ in the relation above and solving for $x^*(t_f)$, which gives

$$x^*(t_f) = \frac{2\alpha x_0 e^{\alpha t_f}}{2\alpha + \beta^2 \gamma \left(e^{2\alpha t_f} - 1\right)}$$

If we let $\gamma \to \infty$, $x^*(t_f) \to 0$. This is expected, since in this case the cost function J to be minimized has a large weight on the terminal cost and the optimal control signal has to take the particle to a velocity close to zero at $t = t_f$.