Modelling in three phases:

1. Problem structure
   - Formulate purpose, requirements for accuracy
   - Break up into subsystems — What is important?

2. Basic equations
   - Write down the relevant physical laws
   - Collect experimental data
   - Test hypotheses
   - Validate the model against fresh data

3. Model with desired features is formed
   - Put the model on suitable form.
     (Computer simulation or pedagogical insight?)
   - Document and illustrate the model
   - Evaluate the model: Does it meet its purpose?

Outline

Thursday lecture
- Course introduction
- Ethics of modelling
- Static models from data (black boxes)

Friday lecture
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- Models from physics (white boxes)
- Mixed models (grey boxes)

Basic idea of system identification

Measure \( U \) and \( y \). Figure out a model of \( S \), consistent with measured data.

Important aspects:
- We can only measure the \( u \) and \( y \) in discrete time points (sampling).
- Can be natural to use the discrete-time models.
- The system is affected by interference and measurement errors. We may need to signal models for dealing with this.

Example

A tank which attenuates flow variations in \( q_1 \). Characterization of the tank system:

\[ \begin{align*}
\text{Input: } q_1 \\
\text{Output: } q_2 \text{ and/or } h \\
\text{Internal variables / conditions: } h
\end{align*} \]

Step response

Can give idea of the dominant time constant, static reinforcement, character (overshoot or not)

Frequency response

For good signal-to-noise ratio, an estimate of \( G(i\omega) \) is obtained directly from the amplitudes and phase positions of \( u, y \)

\[ \begin{align*}
u(t) &= A \sin \omega t \\
y(t) &= A[G(i\omega)] \sin(\omega t + \arg G(i\omega))
\end{align*} \]
How light affects pupil area

Bode-diagram for pupil

Correlation analysis

Can we estimate the impulse response with other inputs?

- Impulse response formula in discrete time ($T = 1, v = \text{noise}$):
  \[
y(t) = \sum_{k=1}^{\infty} g_k u(t-k) + v(t)
  \]

- If $v$ white noise with $E[v^2] = 1$, then
  \[R_{yu}(k) = E[y(t)u(t-k)] = g_k\]

- Covariance $R_{yu}$ estimated by $N$ data points with
  \[
  \hat{R}_{yu}(k) = \frac{1}{N} \sum_{t=1}^{N} y(t)u(t-k)
  \]

Estimated and actual impulse responses

Basic rules

Make experiments with conditions similar to the conditions in which the model is to be used!

(Models from step response can be expected to work best on the stage.)

Save some data for model validation, i.e. check the model with data set different from the one that generated the model!

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Principles and analogies: Hydraulics

Example 1. A hydraulic system:

Incompressible fluid. Pressures: $p_a$, $p_1$, $p_2$, and $p_4$.
Volume flows: $Q_1$, $Q_2$, $Q_3$, and $Q_5$. 

Example

Correlation analysis for $\frac{1}{s^2+2s+1}$ (in- and out-put data)
### Principles and analogies: Electrics

Example 2. An electrical system:

- **Potentials**: $v_a, v_b, v_1, v_2$
- **Currents**: $i_1, i_2, i_3, i_4, i_5$
- **Spring constants**: $k_1, k_2$
- **Damping constants**: $d_1, d_2$
- **Temperatures**: $T_a, T_b, T_1, T_2$

### Principles and analogies: Mechanics

Example 4. A mechanical system:

- **External forces**: $F_a, F_b$
- **Velocities**: $v_1, v_2, v_3$
- **Spring constants**: $k_1, k_2$
- **Damping constants**: $d_1, d_2$

### Principles and analogies: Heat

Example 3. A thermal system

- **Temperatures**: $T_a, T_b, T_1, T_2$
- **Heat flows**: $q_1, q_2, q_3, q_4$

### Analogies (cont’d)

- **Losses**
  - **Flow**: $L \cdot \frac{d}{dt}(flow) = intensity$
  - **Capacitance**: $C \cdot \frac{d}{dt}(intensity) = flow$
  - **Inductance**: $L \cdot \frac{d}{dt}(flow) = intensity$

### Analogies (cont’d)

- **Energy flows**
  - **Pressure**: $p \cdot \text{flow} = \text{power}$
  - **Voltage difference**: $v \cdot \text{current} = \text{power}$
  - **Force**: $F \cdot \text{velocity} = \text{power}$
  - **Torque**: $T \cdot \text{angular velocity} = \text{power}$
  - **Temperature**: $T \cdot \text{heat flow} = \text{power} - \text{temperature}$

---

**Flow Variables**
- Temperature
- Force
- Pressure
- Power flow
- Voltage
- Speed
- Heat flow

**Intensity variables**
- External forces: $F$
- Velocities: $v$
- Spring constants: $k$
- Damping constants: $d$
- Temperatures: $T$
- Hydraulics: $\rho l$
- Electrics: $\text{inductans}$
- Mechanics: $m$
- Heat: $\text{thermal capacity}$
- Electric: $\text{resistance}$
- Hydraulic: $\text{flow resistance}$

**Balance equations!**

- Mechanics: mass
- Heat: thermal capacity
- Electric: inductans
- Hydraulic: $\rho l/A$

**Often linear relationship in the electrical case - nonlinearly in the other**

(may be approximated by linear for small changes of variable)

(more complicated if the capacitance is not constant.)
**Dimension analysis**

Physical variables have dimensions. E.g.,

\[
\text{[density]} = ML^{-3}
\]

\[
\text{[force]} = M \cdot \frac{L}{T_2} = MLT^{-2}
\]

where

\(M = \text{[mass]}, \ T = \text{[time]}, \ L = \text{[length]}\)

Physical connections must be dimensionally “correct”.

---

**Example: Bernoulli’s law**

In Bernoulli’s law \(v = \sqrt{\frac{2gH}{\mathcal{h}}}\) you have

\[
\left[\frac{v}{\sqrt{\mathcal{h}}}\right] = LT^{-1}(LT^{-2}L)^{-0.5} = 1
\]

\(v/\sqrt{\mathcal{h}}\) is an example of dimensionless quantity.

---

**Dimensionless quantities and scaling**

Some historical passenger ships:

- Kaiser Wilhelm the great, 1898, 22 knots, 200 m
- Lusitania, 1909, 25 knots, 240 m
- Rex, 1933, 27 knots, 269 m
- Queen Mary, 1938, 29 knots, 311 m

Note that the ratio \((\text{velocity})^2/\text{(length)}\) is almost constant

Which physical phenomenon can be thought to be the cause?

---

**2 min problem**

Find the relationship (except for a scaling by a dimensionless constant) between a pendulum period time and its mass, its length and the acceleration of gravity \(g\), i.e.,

\[t = f(m, l, g)\]

---

**A Graphical Modelica Model**

---

**A submodel can be opened**

---

**Simple system in text format**

```model FirstOrder
Real x;
equation
  der(x) = 1-x;
end FirstOrder;
```

Documentation is important:

```model FirstOrderDocumented
  "A simple first order differential equation"
  Real x "State variable";
equation
  der(x) = 1-x "Drives value of x toward 1.0";
end FirstOrderDocumented;
```

---

**Initialize at equilibrium!**

```model FirstOrderSteady
  "First order equation with steady state initial condition"
  Real x "State variable";
initial equation
  der(x) = 0 "Initialize the system in steady state";
equation
  der(x) = 1-x "Drives value of x toward 1.0";
end FirstOrderSteady;
```
Nonlinear differential-algebraic equations (DAE)

Differential-algebraic equations, DAE

\[ F(z, z, u) = 0, \quad y = H(z, u) \]

\( u \): input, \( y \): output, \( z \): "Internal variable"

Special case: state model

\[ x = f(x, u), \quad y = h(x, u) \]

\( u \): input, \( y \): output, \( x \): state

Linear differential-algebraic equations (DAE)

\[ E \dot{z} = Fz + Gu \]

If \( E \) were non-singular, one could write

\[ z = E^{-1} Fz + E^{-1} Gu \]

which is a valid state model. If \( E \) is singular, variables have to be eliminated to get a state equation. Using a DAE solver is often better, since elimination can destroy sparsity.

Example:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & J_2 & 0 & 0 \\
0 & 0 & 0 & J_2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\phi_2 \\
\omega_1 \\
\omega_2
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\omega_1 \\
\omega_2
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_3 \\
\tau_4
\end{bmatrix}
\]

Mathematics of general connection:

State models for two separate components:

\[
\dot{\phi}_1 = \omega_1, \quad \phi_2 = \omega_2, \quad J_{1\omega_1} = \tau_1 + \tau_2, \quad J_{2\omega_2} = \tau_3 + \tau_4
\]

Connection:

\[
\phi_1 = \phi_2, \quad \tau_2 = -\tau_3
\]

The resulting model is not exactly a state model.

Prediction Error Method with Repeated Simulation

For a nonlinear grey-box model

\[
0 = F(x, x, t, \theta) \\
y(t) = h(x, t, \theta)
\]

the unknown parameters \( \theta \) could be determined by the prediction error method

\[
\min_{\theta} \| \hat{y}(t, \theta) - y(t) \|
\]

where the output prediction \( \hat{y}(t, \theta) \) is computed by simulation.

(Repeated simulation for different values of \( \theta \) could however be very time-consuming.)
Population dynamics / Ecology
Variations in the number of lynx (solid) and hares (dashed) in Canada. Can you predict the periodic variations?

\[
\begin{align*}
\frac{d}{dt} N_1(t) &= (\lambda_1 - \gamma_1) N_1(t) + \alpha_1 N_1(t) N_2(t) \\
\frac{d}{dt} N_2(t) &= (\lambda_2 - \gamma_2) N_1(t) - \alpha_2 N_1(t) N_2(t)
\end{align*}
\]

Simulation:

Mixing tanks in Skärblacka paper factory
A linear transfer function of three series-connected mixing tanks has the form \( \frac{1}{s^3 + s + 1} \).
To determine \( \theta \), radioactive lithium is added in A. Radioactivity was then measured by B as a function of time.

Impulse response
In the lower picture, \( \theta \) has been chosen to adapt to the impulse response of \( \frac{1}{s^3 + s + 1} \).

Grey Models — the best of both worlds
◮ White boxes: Physical laws provide some insight
◮ Black boxes: Statistics estimates complex relationships
◮ Gray boxes: Combine simplicity with insight

My Own Research: Dynamic Buffer Networks
◮ Producers, consumers and storages
◮ Examples: water, power, traffic, data
◮ Discrete/continuous, stochastic/deterministic
◮ Multiple commodities, human interaction
Problem: Scalable and adaptive methods for control.

Example: Heating Networks
Grey box modelling!

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Good luck with your projects!
I look forward to receiving your project plans.