Predictive Control – Exercise Session 3

Optimal Prediction, Optimal Estimation, Kalman Filter

1. Consider the discrete-time state space model of a DC motor:

$$x_{k+1} = \begin{pmatrix} 0.78 & 0 \\ 0.22 & 1 \end{pmatrix} x_k + \begin{pmatrix} 0.22 \\ 0.03 \end{pmatrix} u_k$$
$$y_k = \begin{pmatrix} 0 & 1 \end{pmatrix} x_k$$

with sampling interval $h = 0.25$.

a. Determine an observer on the form

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + K\left(y_k - C\hat{x}_{k|k-1}\right).$$

such that the observer poles are placed in the origin (this is known as a 'deadbeat observer').

b. The observer in (a) has a delay of one sample, because $\hat{x}_k$ depends only on measurements up to time $k-1$. The following observer can be used to avoid the delay:

$$\begin{aligned}\hat{x}_k &= A\hat{x}_{k-1} + Bu_{k-1} + K\left(y_k - C(A\hat{x}_{k-1} + Bu_{k-1}\right) \\ &= (I - KC)\left(A\hat{x}_{k-1} + Bu_{k-1}\right) + Ky_k.\end{aligned}$$

Determine an observer on this form such that the observer poles are placed in the origin.

2. A stochastic process is generated as

$$\begin{aligned}x_{k+1} &= 0.5x_k + v_k \\ y_k &= x_k + e_k\end{aligned}$$

where v and e are uncorrelated white-noise processes with the variances r_1 and r_2, respectively. Further, x_0 is normally distributed with zero mean and variance r_0.

a. Determine the Kalman filter (predictor case, i.e., without direct term) for the system.

b. What is the variance of the estimation error (P) and the filter gain (K) in steady state?

c. What happens to the filter in the special cases $r_1 \gg r_2$ and $r_1 \ll r_2$? Compute the filter gain (K) and the pole of the filter ($A - KC$) in both cases.

3. Consider optimal prediction of the process

$$y_{k+1} \quad = \quad 1.5y_k - 0.9y_{k-1} + w_{k+1} + 0.7w_k \qquad (1)$$

where $\{w_k\}$ is a zero-mean uncorrelated stochastic process with variance $\mathcal{E}\{w_k w_j^T\} = \sigma_w^2 \delta_{kj}$.

a. Calculate a one-step predictor for the process of (1) and the resulting prediction covariance.

b. Calculate a two-step predictor for the process of (1) and the resulting prediction covariance.

4. Consider minimum-variance control of the process

$$y_{k+1} = 1.5y_k - 0.9y_{k-1} + u_k + 0.9u_{k-1} + w_{k+1} + 0.7w_k \qquad (2)$$

where $\{w_k\}$ is a zero-mean uncorrelated stochastic process with variance $\mathcal{E}\{w_k w_j^T\} = \sigma_w^2 \delta_{kj}$.

a. Calculate the one-step-ahead minimum-variance control for the process of (2) and the resulting prediction covariance.

b. Calculate a two-step-ahead minimum-variance control for the process of (2) and the resulting prediction covariance.

c. Calculate a one-step-ahead minimum-variance control for the process of (2) without cancellation of the B-polynomial. Calculate the resulting prediction covariance.