Lecture 6

FRTN10 Multivariable Control

Automatic Control LTH, 2019

Course Outline

L1–L5 Specifications, models and loop-shaping by hand

L6–L8 Limitations on achievable performance

6. **Controllability/observability, multivariable poles/zeros**
7. Fundamental limitations
8. Multivariable and decentralized control

L9–L11 Controller optimization: analytic approach

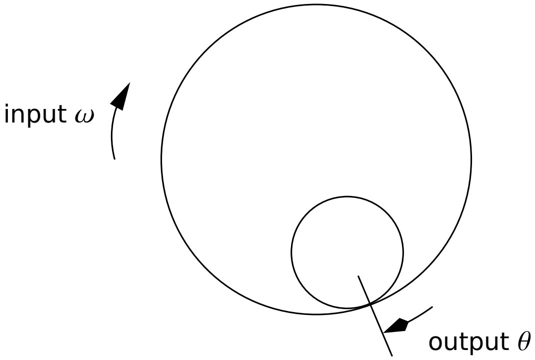
L12–L14 Controller optimization: numerical approach

L15 Course review

L6: Controllability/observability, multivariable poles/zeros

1. Controllability and observability, Gramians
2. Multivariable poles and zeros
3. Minimal realizations

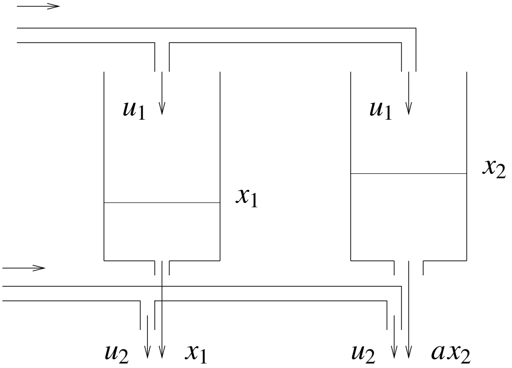
Example: Ball in the Hoop



$$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$$

Can you reach $\theta = \pi/4$, $\dot{\theta} = 0$? Can you stay there?

Example: Two water tanks



$a \geq 1$

$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$
$$\dot{x}_2 = -ax_2 + u_1 \qquad y_2 = ax_2 + u_2$$

Can you reach $y_1 = 1, y_2 = 2$? Can you stay there?

L6: Controllability/observability, multivariable poles/zeros

Controllability – definition

The system

$$\dot{x} = Ax + Bu$$

is **controllable**, if for every $x_1 \in \mathbb{R}^n$ there exists $u(t)$, $t \in [0, t_1]$, such that $x(t_1) = x_1$ can be reached from $x(0) = 0$.

The collection of vectors x_1 that can be reached in this way is called the **controllable subspace** and is given by the range of the **controllability matrix**

$$\mathcal{C} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

Controllability criteria

The following controllability criteria for a system $\dot{x} = Ax + Bu$ of order n are equivalent:

(i) rank $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$

(ii) rank $\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$

If the system is stable, define the **controllability Gramian**

$$W_c = \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

For such systems there is a third equivalent criterion:

(iii) The controllability Gramian is non-singular

Interpretation of controllability Gramian

Let $x(0) = 0$ and

$$u(t) = \begin{bmatrix} \delta(t) & \dots & \delta(t) \end{bmatrix}^T$$

Then the state will move as

$$x(t) = e^{At} B$$

Amount of "energy" in the different states:

$$\int_0^\infty x(t)x^T(t)dt = \int_0^\infty e^{At} B B^T e^{A^T t} dt = W_c$$

Furthermore, the control energy required to reach a given state $x = x_1$ starting from $x = 0$ satisfies

$$\int_0^\infty |u(t)|^2 dt \geq x_1^T W_c^{-1} x_1$$

(For proof, see the lecture notes.)

Computing the controllability Gramian

The controllability Gramian $W_c = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ can be computed by solving the Lyapunov equation

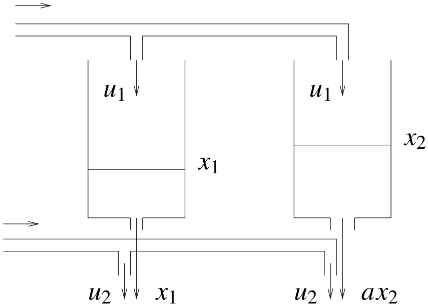
$$A W_c + W_c A^T + B B^T = 0$$

(For proof, see the lecture notes.)

(Matlab: `Wc = lyap(A,B*B')`)

(Q: Where have we seen this equation before?)

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad\qquad \dot{x}_2 = -ax_2 + u_1$$

Controllability Gramian: $W_c = \int_0^\infty \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$

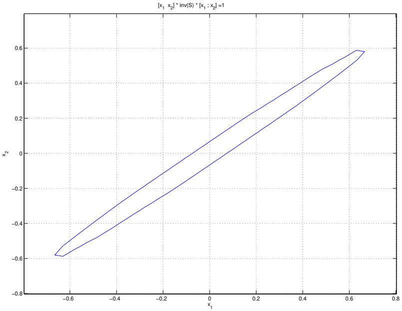
W_c close to singular when $a \approx 1$.

Example cont'd

Matlab:

```
>> a = 1.25;   A = [-1 0; 0 -1*a];   B = [1; 1];
```

```
>> CM = [B  A*B], rank(CM)
CM =
    1.0000   -1.0000
    1.0000   -1.2500
ans =
     2
>> Wc = lyap(A,B*B')
Wc =
   0.5000    0.4444
   0.4444    0.4000
>> invWc = inv(Wc)
invWc =
    162.0   -180.0
   -180.0    202.5
```



Plot of $\begin{bmatrix} x_1 & x_2 \end{bmatrix} W_c^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$

corresponds to the states we can reach by $\int_0^\infty |u(t)|^2 dt = 1$.

Observability – definition

The system

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

is **observable**, if the initial state $x(0) = x_0 \in \mathbb{R}^n$ can be uniquely determined by the output $y(t)$, $t \in [0, t_1]$.

The collection of vectors x_0 that cannot be distinguished from $x = 0$ is called the **unobservable subspace** and is given by the nullspace of the **observability matrix**

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Observability criteria

The following observability criteria for a system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ of order n are equivalent:

(i) rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

(ii) rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$

If the system is stable, define the **observability Gramian**

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

For such systems there is a third equivalent statement:

(iii) The observability Gramian is non-singular

Interpretation of observability Gramian

Let $x(0) = x_0$. Then the state will move as

$$x(t) = e^{At} x_0$$

Amount of energy in the output $y = Cx$:

$$\int_0^\infty |y(t)|^2 dt = \int_0^\infty x^T(t) C^T C x(t) dt = x_0^T \underbrace{\int_0^\infty e^{A^T t} C^T C e^{At} dt}_{W_o} x_0$$

The observability Gramian measures how easy it is to distinguish an initial state from zero by observing the output.

Computing the observability Gramian

The observability Gramian $W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ can be computed by solving the Lyapunov equation

$$A^T W_o + W_o A + C^T C = 0$$

(Matlab: `Wo = lyap(A',C'*C)`)

Mini-problem

Two water tanks:

$$\dot{x}_1 = -x_1 \qquad\qquad y_1 = x_1$$
$$\dot{x}_2 = -ax_2 \qquad\qquad y_2 = ax_2$$

Is the water tank system with $a = 1$ observable?

What if only y_1 is available?

1. Controllability and observability
2. Multivariable poles and zeros
3. Minimal realizations

Poles and zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{G(s)} U(s)$$

For **scalar** systems,

- the points $p \in \mathbb{C}$ where $G(p) = \infty$ are called **poles**
- the points $z \in \mathbb{C}$ where $G(z) = 0$ are called **zeros**

Poles and zeros

For multivariable systems,

- the points $p \in \mathbb{C}$ where any $G_{ij}(p) = \infty$ are called **poles**
- the points $z \in \mathbb{C}$ where $G(z)$ loses rank are called **(transmission/multivariable) zeros**

Example:

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: -2 and -1 (but what about their multiplicity?)

Zeros: 1 (but how to find them?)

Poles and zeros

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Poles: -2 and -1 (but what about their multiplicity?)

Zeros: 1 (but how to find them?)

Pole and zero polynomials

- The **pole polynomial** is the least common denominator of all minors* of $G(s)$.
- The **zero polynomial** is the greatest common divisor of the maximal minors of $G(s)$, normalized to the have the pole polynomial as denominator.

The **poles** of G are the roots of the pole polynomial.

The **zeros** of G are the roots of the zero polynomial.

* A minor of a matrix A is the determinant of some square submatrix, obtained by removing zero or more of A's rows and columns

Poles and zeros – example

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: Minors: $\frac{2}{s+1}, \frac{3}{s+2}, \frac{1}{s+1}, \frac{1}{s+1}, \frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{-(s-1)}{(s+1)^2(s+2)}$

The least common denominator is $(s+1)^2(s+2)$, giving the poles -2 (with multiplicity 1) and -1 (with multiplicity 2)

Zeros: Maximal (2×2) minor: $\frac{-(s-1)}{(s+1)^2(s+2)}$ (already normalized)

The greatest common divisor is $s-1$, giving the (transmission) zero 1 (with multiplicity 1)

(Matlab: `tzero(G)`)

Poles and zeros – example

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: Minors: $\frac{2}{s+1}$, $\frac{3}{s+2}$, $\frac{1}{s+1}$, $\frac{1}{s+1}$, $\frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{-(s-1)}{(s+1)^2(s+2)}$

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Poles and zeros – example

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(Matlab: `tzero(G)`)

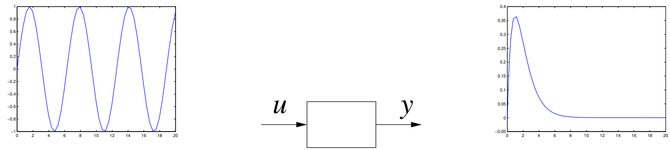
Interpretation of poles and zeros

Poles:

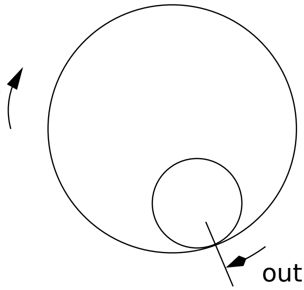
- A pole p is associated with the state response $x(t) = x_0 e^{pt}$
- A pole p is an eigenvalue of A

Zeros:

- A zero z means that an input $u(t) = u_0 e^{zt}$ is blocked
  - For a multivariable system, blocking occurs only in a certain input direction
- A zero describes how inputs and outputs couple to states



Example: Ball in the Hoop

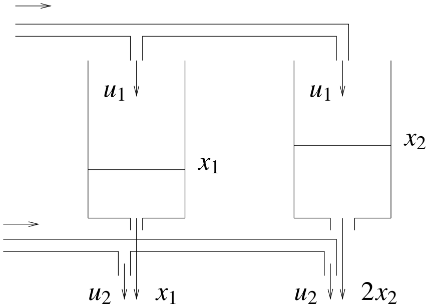


$$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$$

The transfer function from ω to θ is $\frac{s}{s^2+cs+k}$. The zero in $s = 0$ makes it impossible to control the stationary position of the ball.

- Zeros are not affected by feedback!

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$
$$\dot{x}_2 = -2x_2 + u_1 \qquad y_2 = 2x_2 + u_2$$

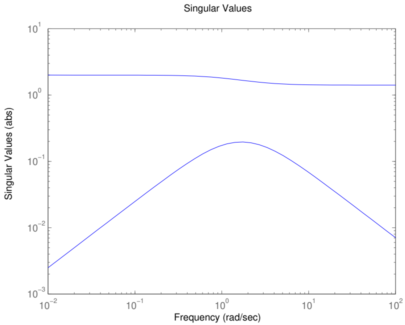
$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix} \qquad \det G(s) = \frac{-s}{(s+1)(s+2)}$$

This system also has a zero in the origin! At stationarity $y_1 = y_2$.

Plot singular values of $G(i\omega)$ vs frequency

```
» s=tf('s')
» G=[1/(s+1) 1 ; 2/(s+2) 1]
» sigma(G); plot singular values

% Alt. for a certain frequency:
» w = 1;
» A = evalfr(G,i*w);
» [U,S,V] = svd(A)
```



The largest singular value of $G(i\omega) = \begin{bmatrix} \frac{1}{i\omega+1} & 1 \\ \frac{2}{i\omega+2} & 1 \end{bmatrix}$ is fairly constant.

This is due to the second input. The first input makes it possible to control the difference between the two tanks, but mainly near $\omega = 1$ where the dynamics make a difference.

L6: Controllability/observability, multivariable poles/zeros

Minimal realization – definition

Given $G(s)$, any state-space model (A, B, C, D) that is both **controllable** and **observable** and has the same input–output behavior as $G(s)$ is called a **minimal realization**.

A transfer function with n poles (counting multiplicity) has a minimal realization of order n.

Realization in diagonal (modal) form

Consider a transfer function with partial fraction expansion

$$G(s) \quad = \quad \sum_{i=1}^{n} \frac{C_i B_i}{s - p_i} + D$$

This has the realization

$$\dot{x}(t) = \begin{bmatrix} p_1 I & & 0 \\ & \ddots & \\ 0 & & p_n I \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix} x(t) + Du(t)$$

The rank of the matrix $C_i B_i$ determines the necessary number of rows in B_i, columns in C_i, and the multiplicity of the pole p_i.

Realization of multivariable system – example 1

To find a minimal realization for the system

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

with poles in -2 and -1 (double), write the transfer matrix as (e.g.)

$$G(s) = \frac{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+2}$$

giving the realization

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix} x$$

Realization of multivariable system – example 2

To find state space-realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+3)} \\ \frac{6}{(s+2)(s+4)} & \frac{1}{s+2} \end{bmatrix}$$

write the transfer matrix as

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+3} \\ \frac{3}{s+2} - \frac{3}{s+4} & \frac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}}{s+1} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}3 & 1\end{bmatrix}}{s+2} + \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & -1\end{bmatrix}}{s+3} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}-3 & 0\end{bmatrix}}{s+4}$$

This gives the realization

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x$$

Lecture 6 – summary

- Gramians give quantitative answers to how controllable or observable a system is in different state directions
  - Warning: They do not reveal some important frequency-domain information (see next lecture)
- A multivariable zero blocks input signals in a certain direction
  - A zero in the origin makes it impossible to control the system in stationarity
- A minimal state-space realization describes the controllable and observable subspace of a system