LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam – January 13, 2020, 08:00 – 13:00

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem.

Grade limits:

Grade 3: 10 – 16 points
4: 16.5 – 21 points
5: 21.5 – 25 points

Accepted aid

All course material, except for exercises, old exams, lab instructions, and solutions of these, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

Note!
In many cases the subproblems can be solved independently of each other.

Good luck!

1. Find all equilibrium points of the following system and classify them, e.g. unstable node/focus etc.

$$\begin{aligned}\dot{x}_1 &= x_2 \sin(\pi x_1)\\ \dot{x}_2 &= 2x_1 - \cos(\pi x_2)\end{aligned}$$

(3 p)

Solution

The equilibrium points are those (x_1, x_2) such that $\dot{x}_1 = \dot{x}_2 = 0$, i.e.

$$\begin{aligned}0 = \dot{x}_1 &= x_2 \sin(\pi x_1)\\ 0 = \dot{x}_2 &= 2x_1 - \cos(\pi x_2)\end{aligned}$$

From the first equation we see that either $x_2 = 0$ or $x_1 = n$, where n is any integer. In the case $x_2 = 0$, the second equation gives $x_1 = \frac{1}{2}$. In other words, $(\frac{1}{2}, 0)$ is an equilibrium point.

As for the second case $x_1 = n$, since $-1 \leq \cos(t) \leq 1$ the cosine term in the second equation cannot match the x_1 term if $n \neq 0$. Hence, the only possibility is that $n = 0$ and so $x_1 = 0$. But this means $\cos(\pi x_2) = 0$ which is possible if and only if $x_2 = \frac{1}{2} + k$ for any integer k.

Equilibrium points: $\{(\frac{1}{2}, 0), (0, \frac{1}{2} + k)\}$

In order to classify the equilibrium points, we consider the system matrix A for the linearized system evaluated at these points and find its eigenvalues.

$$A(x_1, x_2) = \frac{\partial f}{\partial x} = \begin{pmatrix} \pi x_2 \cos(\pi x_1) & \sin(\pi x_1) \\ 2 & \pi \sin(\pi x_2) \end{pmatrix}$$

Evaluating the system matrix at each equilibrium point, we have

$$A(\frac{1}{2}, 0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$A(0, \frac{1}{2} + k) = \begin{pmatrix} \pi k + \frac{\pi}{2} & 0 \\ 2 & (-1)^k \pi \end{pmatrix}$$

The matrix corresponding to the point $(\frac{1}{2}, 0)$ has the characteristic polynomial $s^2 - 2$ and hence eigenvalues $\lambda = \pm\sqrt{2}$. The equilibrium point $(\frac{1}{2}, 0)$ therefore corresponds to a saddle point.

As for the other matrix, due to its triangular shape the eigenvalues are visible on the diagonal. One eigenvalue alternates between a positive value for even k and a negative value for odd k. Additionally, for non-negative k, the other eigenvalue is always positive whereas it is always negative for negative k. Thus, the equilibrium point $(0, \frac{1}{2} + k)$ for non-negative k corresponds to an unstable node for even k and saddle point for odd k, whereas for negative k it corresponds to a saddle point for even k and a stable node for odd k.

2. The motion of a satellite orbiting around Earth can be modeled in the plane by the following equations

$$\ddot{r}(t) = r(t)\dot{\theta}^2(t) - \frac{\alpha}{r^2(t)} + u_1(t)$$
$$\ddot{\theta}(t) = \frac{-2\dot{r}(t)\dot{\theta}(t)}{r(t)} + \frac{u_2(t)}{r(t)}$$

Here, r denotes the distance from the center of Earth to the satellite and θ the angle with respect to some axis. Further, the satellite is equipped with the means of applying forces u_1 and u_2 in the radial and tangential directions respectively. α is a constant.

a. Introduce the state variables $x_1(t) = r(t)$, $x_2(t) = \dot{r}(t)$, $x_3(t) = \theta(t)$ and $x_4(t) = \dot{\theta}(t)$, and express the system on state-space form. (1 p)

b. Verify that a solution $x^*(t)$ to the system when the input is set to zero, i.e. $u_1^*(t) = u_2^*(t) = 0$, is given by

$$x_1^*(t) = c, \quad x_2^*(t) = 0, \quad x_3^*(t) = pt + q, \quad x_4^*(t) = p$$

Here, c, p, q are constants and $p = \sqrt{\frac{\alpha}{c^3}}$. What kind of motion does the solution represent? (1 p)

c. Linearize around the trajectory $(x^*(t), u^*(t))$. (1 p)

d. Is the trajectory locally asymptotically stable? (1 p)

Solution

a. With the introduced state variables, the system may be expressed as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 x_4^2 - \frac{\alpha}{x_1^2} + u_1$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{-2x_2x_4}{x_1} + \frac{u_2}{x_1}$$

b. Plugging the given expressions into the equations of motion with $u^*(t) = 0$ gives

$$\dot{x}_1^* = \dot{c} = 0 = x_2^*$$
$$\dot{x}_2^* = \dot{0} = 0 = c\frac{\alpha}{c^3} - \frac{\alpha}{c^2} = x_1^* x_4^{*2} - \frac{\alpha}{x_1^{*2}}$$
$$\dot{x}_3^* = \frac{d}{dt}(pt + q) = p = x_4^*$$
$$\dot{x}_4^* = \dot{p} = 0 = \frac{-2 \cdot 0 \cdot p}{c} = \frac{-2x_2^* x_4^*}{x_1^*}$$

This shows that $x^*(t)$ satisfies the equations and is thus a solution.

The solution represents a circular trajectory (constant $c = x_1(t) = r(t)$) with constant angular velocity (constant $p = x_4 = \dot{\theta}$) in the plane.

c. We begin by formulating the Jacobians for the system.

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + 2\frac{\alpha}{x_1^3} & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ \frac{2x_2x_4}{x_1^2} - \frac{u_2}{x_1^2} & \frac{-2x_4}{x_1} & 0 & \frac{-2x_2}{x_1} \end{pmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{x_1} \end{pmatrix}$$

Evaluating the matrices along $(x^*(t), u^*(t))$, we have

$$A = \frac{\partial f}{\partial x}(x^*(t), u^*(t)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3p^2 & 0 & 0 & 2cp \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-2p}{c} & 0 & 0 \end{pmatrix}$$

$$B = \frac{\partial f}{\partial u}(x^*(t), u^*(t)) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{c} \end{pmatrix}$$

With the deviation variables $\tilde{x}(t) = x(t) - x^*(t)$ and $\tilde{u} = u(t) - u^*(t) = u(t)$, the linearized system finally becomes

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

Note that A and B are time-invariant, an atypical occurrence given that the linearization normally changes depending on the point at which the linearization is made.

d. The solution is not locally asymptotically stable. If it were, then by definition there would exist a $k > 0$ such that any solution starting within distance k of $x^*(0)$ would also have to converge to $x^*(t)$. But given such a k, one can always take the solution $x^{**}(t)$ as identical to $x^*(t)$ except for $x_3^{**}(t) = x_3^*(t) - \frac{k}{2}$. This new solution $x^{**}(t)$ (movement along the same circle, but starting at a different point) starts within distance k of $x^*(0)$ but does not converge to $x^*(t)$, a contradiction.

3. Consider the following system

$$\begin{aligned} \dot{x}_1 &= ax_1 - x_2^3 + \mu(x) \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

where $\mu(x)$ denotes a control law and a is a constant.

a. Use exact linearization to find a $\mu(x)$ such that the origin becomes globally asymptotically stable. (2 p)

b. Suppose now that we replace a with an additional state x_3 which we can influence directly according to

$$\begin{aligned}\dot{x}_1 &= x_1x_3 - x_2^3 \\ \dot{x}_2 &= x_1 - x_2 \\ \dot{x}_3 &= \mu(x)\end{aligned}$$

Find a $\mu(x)$ such that the origin becomes globally asymptotically stable. (2 p)
Hint: Consider the Lyapunov function candidate

$$V(x) = \frac{x_1^2}{2} + \frac{x_2^4}{4} + \frac{x_3^2}{2}$$

Solution

a. Choose $\mu(x)$ in such a way that the nonlinearity is removed, the system becomes linear and the system matrix has LHP eigenvalues. This can be accomplished in many different ways, and one suggestion is

$$\mu(x) = x_2^3 - (a+1)x_1$$

Plugging this μ into the system equations, we have

$$\begin{aligned}\dot{x}_1 &= ax_1 - x_2^3 + \mu = ax_1 - x_2^3 + (x_2^3 - (a+1)x_1) = -x_1 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

The resulting system can be expressed in matrix form as $\dot{x} = Ax$ with

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

and its triangular shape immediately allows us to see its eigenvalues λ on the diagonal: $\lambda = -1, -1$. Because the eigenvalues are in the strict LHP and the system is linear, it must be globally asymptotically stable.

b. We begin by convincing ourselves that the function provided in the hint really is a Lyapunov function. First, it is clear that $V(x) > 0\ \forall x \neq 0$ with $V(0) = 0$. Then we have

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + x_2^3\dot{x}_2 + x_3\dot{x}_3 \\ &= x_1(x_1x_3 - x_2^3) + x_2^3(x_1 - x_2) + x_3\mu \\ &= x_1^2x_3 - x_1x_2^3 + x_2^3x_1 - x_2^4 + x_3\mu \\ &= x_3(\mu + x_1^2) - x_2^4\end{aligned}$$

In order for V to be a Lyapunov function, it has to satisfy $\dot{V}(x) \leq 0$ along all trajectories $x(t)$. One way of achieving this is by choosing

$$\mu(x) = -x_3 - x_1^2$$

Then

$$\dot{V}(x) = x_3(\mu + x_1^2) - x_2^4 = x_3((-x_3 - x_1^2) + x_1^2) - x_2^4 = -x_3^2 - x_2^4 \leq 0$$

and the system becomes

$$\begin{aligned}\dot{x}_1 &= x_1x_3 - x_2^3\\ \dot{x}_2 &= x_1 - x_2\\ \dot{x}_3 &= -x_3 - x_1^2\end{aligned}$$

Consider now the set

$$E = \{(x_1, x_2, x_3) \mid \dot{V}(x) = 0\} = \{(x_1, x_2, x_3) \mid x_2 = x_3 = 0\}$$

and take a point inside with $x_1 \neq 0$. Since $x_2 = x_3 = 0$ for any element in E, we have $\dot{x}_2 = x_1 - x_2 = x_1 \neq 0$ and so x_2 will either increase or decrease. Hence, the trajectory starting in that point will eventually violate the definition of E and leave the set. With the origin being an equilibrium point, the largest invariant subset of E must therefore be $\{(0, 0, 0)\}$. Finally, since clearly $V(x) \to \infty$ as $\|x\| \to \infty$, LaSalle's invariant set theorem may be invoked to show that the choice $\mu(x) = -x_3 - x_1^2$ makes the origin a globally asymptotically stable equilibrium point.

4. The central bank of Sweden (Riksbanken) has as objective to regulate the inflation $y(t)$ [%] such that it is kept close to a setpoint of $r(t) = 2$ [%]. As control signal to achieve this, they can decide on a certain interest rate $u(t)$ [%], which is used for lending money to banks, the so-called "official bank rate" (reporäntan). Now, we would like to replace the central bank with a proportional controller.

Assume that the relationship between the bank rate $u(t)$ and the inflation $y(t)$ can be described with the following model:

$$G(s) = \frac{384}{(s+4)^2(s+12)}$$

where t is the time in years. We want to decide the bank rate $u(t)$ with the proportional controller $C(s)$.

Now assume that the allowed bank rate is restricted to between -0.25 and 0.25 percent. The closed-loop system then becomes as shown in Fig. 1, with the non-linearity

$$f(x) = \begin{cases} x & \text{if } |x| \leq 0.25 \\ 0.25 & \text{if } x > 0.25 \\ -0.25 & \text{if } x < -0.25 \end{cases}$$

(6 p)

a. Give a formula for the describing function $N(A)$ of the non-linearity $f(x)$.

For the following sub-problems, you may use the Bode plot of $G(s)$ in Fig. 2, as well as the describing function plot in Fig. 3. Describe your reasoning carefully.

b. Predict the amplitude and frequency of possible limit cycles in the closed-loop system in Fig. 1 when $C(s) = 10$. Alternatively, prove that the closed-loop system is BIBO stable.

c. Predict the amplitude and frequency of possible limit cycles in the closed-loop system in Fig. 1 when $C(s) = 1/4$. Alternatively, prove that the closed-loop system is BIBO stable.

d. Predict the amplitude and frequency of possible limit cycles in the closed-loop system in Fig. 1 when $C(s) = 3/4$. Alternatively, prove that the closed-loop system is BIBO stable.

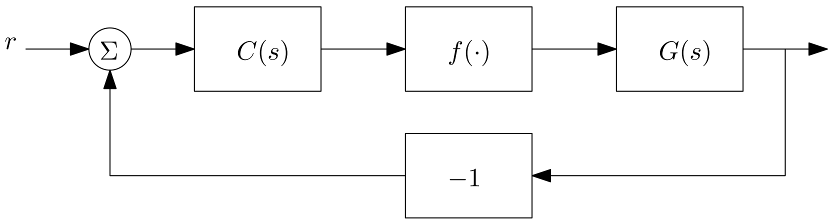

Figure 1: Closed-loop system with limited control signal $u(t)$.

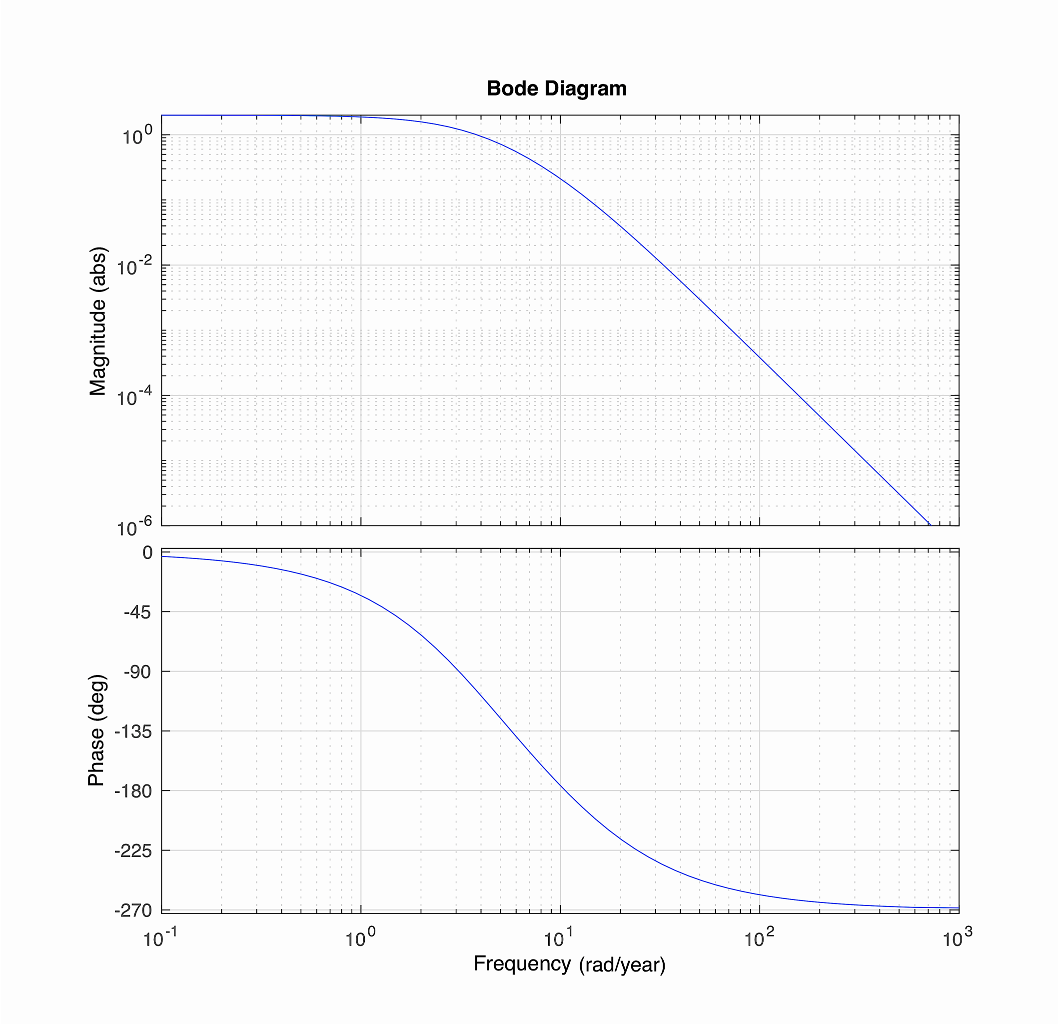

Figure 2: Bode plot for the system $G(s)$.

Solution

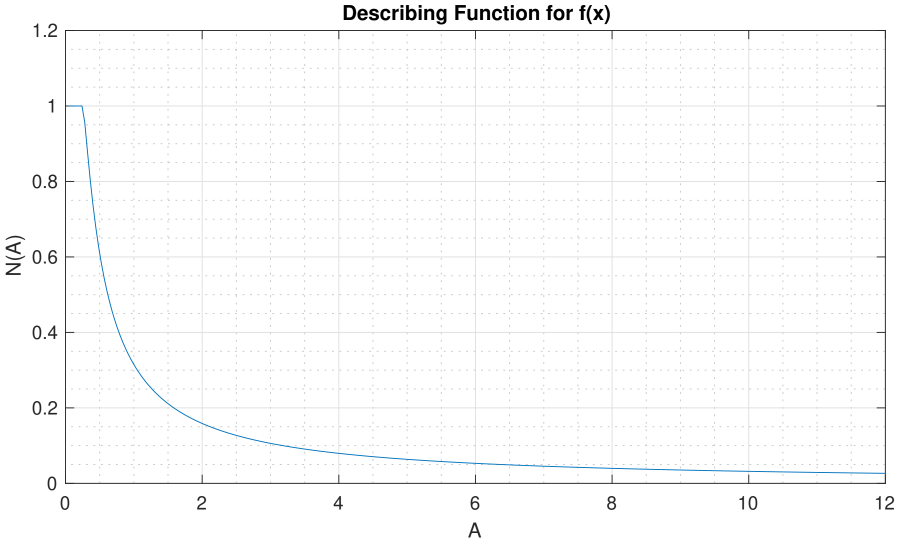

Figure 3: Describing function $N(A)$ for the non-linearity $f(x)$.

a. For a saturation

$$f(x) = \begin{cases} x & \text{if } |x| \leq D \\ D & \text{if } x > D \\ -D & \text{if } x < -D \end{cases}$$

the describing function $N(A)$ is, according to the lecture notes, given by

$$N(A) = \begin{cases} 1 & \text{if } A \leq D \\ \dfrac{1}{\pi}\left(2\phi_0 + \sin\phi_0\right) & \text{if } A > D \end{cases}$$

where $\phi_0 = \arcsin(D/A)$. In this problem, we have $D = 0.25$, so we get

$$N(A) = \begin{cases} 1 & \text{if } A \leq 0.25 \\ \dfrac{1}{\pi}\left(2\phi_0 + \sin\phi_0\right) & \text{if } A > 0.25 \end{cases}$$

with $\phi_0 = \arcsin(0.25/A)$.

b. The system in Fig. 1 can be rewritten as in Fig. 4, where $\tilde{r} = C(s)r$. Describing function analysis predicts a limit cycle with amplitude A and angular frequency ω as the solution of

$$G_0(i\omega) = -\frac{1}{N(A)}, \tag{1}$$

where $G_0(i\omega) := G(i\omega)C(i\omega)$. Since the describing function $N(A)$ is real, possible solutions to this equation must occur when $G_0(i\omega)$ intersects the real axis. As seen in Fig. 3, $N(A)$ takes values between 0 and 1, which means that $-1/N(A)$ takes values in the interval $(-\infty, -1]$. Solutions of equation (1) therefore correspond to intersections of $G_0(i\omega)$ with the real axis in the interval $(-\infty, -1]$. In Fig. 2, we see that we have one such intersection. The phase is $-180°$ for $\omega \approx 10$, which corresponds to an intersection with the negative real axis. At this point, the amplitude of $G(i\omega)$ is $|G(i10)| \approx 0.2$, implying that $|G_0(i10)| = 10|G(i10)| = 2 > 1$, so this is indeed an intersection with the real axis in the interval $(-\infty, -1]$. The amplitude A of the predicted limit cycle is obtained by using the magnitude of equation (1):

$$|N(A)| = \frac{1}{|G_0(i10)|} \approx 1/2.$$

Equation (1) is thus satisfied when $N(A) \approx 1/2$, which in Fig. 3 can be seen to correspond to $A \approx 0.6$. Thus, the describing function analysis predicts a limit cycle with amplitude of about 0.6 and angular frequency of about 10, i.e., with period $T = 2\pi/10 \approx 0.62$.

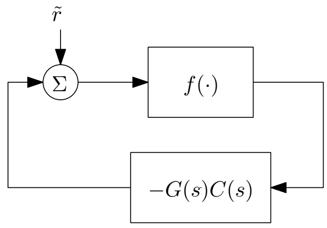

Figure 4: The system in Fig. 1 rewritten, where $\tilde{r} = C(s)r$.

c. In this case, the amplitude of the open-loop system transfer function at the intersection with the negative real axis is $|G_0(i10)| = (1/4)|G(i10)| = 0.05 < 1$, so we do not predict any limit cycles for the closed-loop system with the describing function analysis. We should then instead try to prove that the closed-loop system is BIBO stable. According to the Small Gain Theorem, the closed-loop system in Fig. 4 is stable if

$$\|f\| \cdot \|G_0\| < 1.$$

The gain for f is given by

$$\|f\| = \sup_{x \neq 0} \frac{\|f(x)\|_2}{\|x\|_2} = 1.$$

The gain for the open-loop system is

$$\|G_0\| = \sup_{\omega \in (0,\infty)} |G_0(i\omega)| = \frac{1}{4} \cdot \sup_{\omega \in (0,\infty)} |G(i\omega)| = \frac{1}{4} \cdot 2 = \frac{1}{2} < 1,$$

so we can conclude that the system is BIBO stable according to the Small Gain Theorem.

d. As in the previous subproblem, we have that $|G_0(i10)| = (3/4)|G(i10)| = 0.15 < 1$, so we do not predict any limit cycles for the closed-loop system with the describing function analysis. We should then again try to prove that the closed-loop system is BIBO stable. The gain for the open-loop system is

$$\|G_0\| = \sup_{\omega \in (0,\infty)} |G_0(i\omega)| = \frac{3}{4} \cdot \sup_{\omega \in (0,\infty)} |G(i\omega)| = \frac{3}{4} \cdot 2 = \frac{3}{2} > 1,$$

so we *cannot* conclude that the system is BIBO stable using the Small Gain Theorem. Instead, we try to use the Circle Criterion. The non-linearity can be bounded between two linear functions with non-negative slope, i.e.

$$k_1 x \leq f(x) \leq k_2 x,$$

with $k_1 = 0$ and $k_2 = 1$. Furthermore, the system $G_0(s)$ is stable. If we would have $k_1 > 0$, BIBO stability is proven if the Nyquist curve of $G_0(s)$ does not *intersect or encircle* the circle defined by the points $(-1/k_1, 0)$ and $(-1/k_2, 0)$. The limit case $k_1 = 0$ corresponds to that the point $-1/k_1$ goes to $-\infty$, and the circle then becomes a half-plane through the point $(-1/k_2, 0)$. The resulting criterion is that the Nyquist curve of $G(s)$ must stay to the right of the line $\mathrm{Re}\, s = -1/k_2$. In the Bode diagram in Fig. 2, we can see that when $\arg G(i\omega) > 90°$, it holds that $|G(i\omega)| < 4/3$, which gives $|G_0(i\omega)| < \frac{3}{4} \cdot \frac{4}{3} = 1$, so we get that the Nyquist curve always has an amplitude less than 1 in the left half-plane, and therefore it cannot enter to the left of the line $\mathrm{Re}\, s = -1/k_2 = -1$. BIBO stability is therefore proven by the Circle Criterion.

5. Consider the system in Figure 5, where Δ denotes some unknown nonlinear system (this is often called "multiplicative uncertainty"). The system with $\Delta = 0$ is stable. Some relevant amplitude curves are shown in Figure 6. Use the figures to find a bound γ so that the system is stable for all Δ with gain smaller than γ. (2 p)

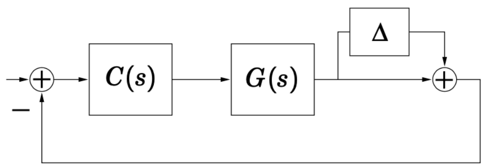

Figure 5: The system in Problem 5.

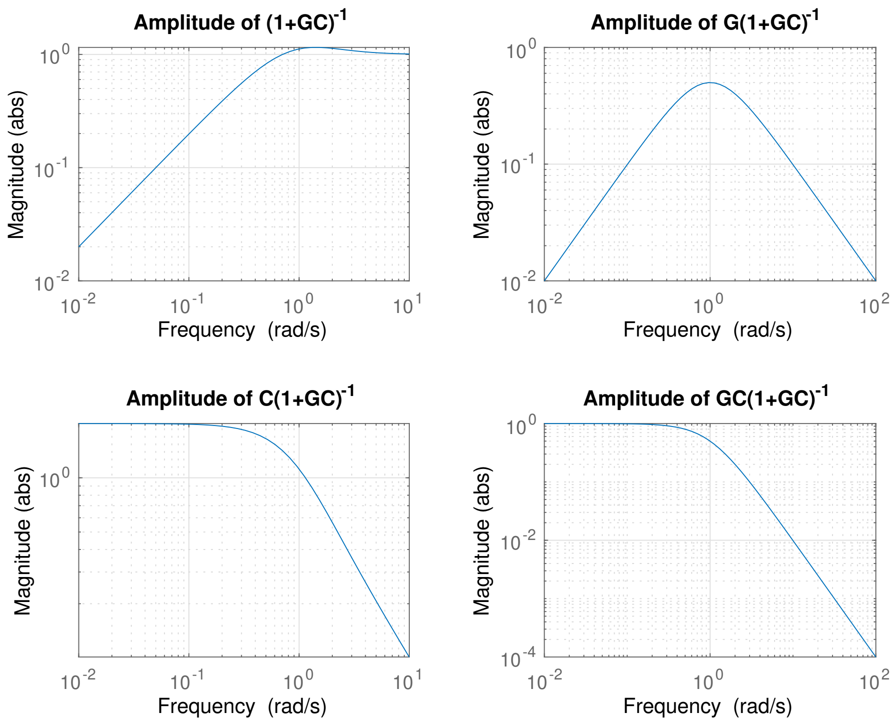

Figure 6: Amplitude curves for Problem 5.

Solution

The diagram can be rewritten as a feedback diagram with $(1 + GC)^{-1}GC$ in the lower box and Δ in the upper. The Small Gain Theorem says that the loop is stable if

$$\|\Delta\| \cdot \|(1 + GC)^{-1}GC\| < 1.$$

From the diagram we read $\|(1 + GC)^{-1}GC\| = 1$, hence we have stability if $\|\Delta\| < 1$. Hence $\gamma = 1$.

6. In this problem we consider the following dynamics

$$\dot{x}_1 = x_2^3 - \arctan x_1 + u$$
$$\dot{x}_2 = x_1$$

a. A friend suggests a sliding set defined by $\sigma(x) = x_1 - x_2$. Show that this suggestion would lead to unstable sliding dynamics. (1 p)

b. Instead you decide to use a sliding mode controller based on $\sigma(x) = x_1 + x_2$. Construct such a controller so that $\sigma(x) = 0$ is a sliding set and all state trajectories reach the sliding set. (2 p)

Solution

a. The sliding set is invariant so $\dot{\sigma}(x)$ should satisfy $\dot{\sigma}(x) = 0$. This gives

$$\dot{\sigma}(x) = \dot{x}_1 - \dot{x}_2 = \dot{x}_1 - x_1 = 0$$

This implies that $x_1 \to \infty$ on the sliding set and the sliding dynamics is hence unstable.

b. We have that

$$\frac{\partial \sigma}{\partial x} f = x_2^3 - \arctan x_1 + u + x_1$$

The controller $u = -x_2^3 + \arctan x_1 - x_1 - \operatorname{sign}(x_1 + x_2)$ will do the job as then for all x (including those satisfying $\sigma(x) = 0$)

$$\frac{\partial \sigma}{\partial x} f_+ = -1$$
$$\frac{\partial \sigma}{\partial x} f_- = 1$$

Thus σ will go to zero, and the sliding surface is a sliding set. One could also define the Lyapunov function $V = \sigma^2/2$ and note that

$$\dot{V} = \sigma\dot{\sigma} = (x_1+x_2)(\dot{x}_1+\dot{x}_2) = (x_1+x_2)(-x_1-\operatorname{sign}(x_1+x_2)+x_1) = -|x_1+x_2|$$

to show that all state trajectories reach the sliding set.

7. Consider the problem

$$\begin{aligned} \text{minimize} \quad & \int_0^{t_f} u^2(t)t \, \mathrm{d}t \\ \text{subject to} \quad & x(0) = c, \quad x(t_f) = 0 \\ & -1 \le u \le 1 \\ & \dot{x} = u \end{aligned}$$

a. Show how the problem can be written as

$$\begin{aligned} \text{minimize} \quad & \int_0^{t_f} u^2(t)x_2(t)\ \mathrm{d}t \\ \text{subject to} \quad & x(0) = [c,\ 0]^T, \quad x(t_f) = [0, t_f]^T \\ & -1 \le u \le 1 \\ & \dot{x}_1 = u \\ & \dot{x}_2 = 1 \end{aligned}$$

(0.5 p)

b. Let $t_f > 1$ and $c > 0$. Show that a solution to the problem in **a** is given by

$$u = \mathrm{sat}(-\mu_1/(2t))$$

for some constant μ_1.
Hint: If $a \ge 0$ then the solution to

$$\begin{aligned} \underset{z}{\text{minimize}} \quad & z^2 a + bz \\ \text{subject to} \quad & -1 \le z \le 1 \end{aligned}$$

is given by $z = \mathrm{sat}(-b/(2a))$. (1.5 p)

c. Verify that μ_1 is given by the solution to

$$c = \frac{\mu_1}{2}\Big(-1 - \log(t_f) + \log(\mu_1/2)\Big).$$

(1 p)

Solution

a. Define $x_1(t) = x(t)$ and introduce a new state $x_2(t) = t$ corresponding to the time. The new state must then have the initial and final conditions $x_2(0) = 0$ and $x_2(t_f) = t_f$. The resulting state dynamics becomes

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= 1 \end{aligned}$$

and we see that we have arrived at the given reformulation.

b. The constraint can be written as $\psi(t_f, x(t_f)) = 0$ with

$$\psi(t_f, x(t_f)) = \begin{bmatrix} x_1(t_f) \\ x_2(t_f) - t_f \end{bmatrix}$$

The Hamiltonian of the problem is given by

$$H = u^2 x_2 + \lambda_1 u + \lambda_2$$

Using that $\dot{\lambda} = -H_x^T$ and $\lambda(t_f)^T = \mu^T \psi_x(t_f, x^*(t_f))$

$$\begin{aligned} \dot{\lambda}_1 &= 0, \quad & \lambda_1(t_f) &= \mu_1 \\ \dot{\lambda}_2 &= u^2 \quad & \lambda_2(t_f) &= \mu_2 \end{aligned}$$

This gives $\lambda_1 = \mu_1$, i.e. a constant. It turns out we do not need to solve for λ_2. The optimal u is the minimizer of H.

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & u^2 x_2 + \lambda_1 u \\ \text{subject to} \quad & -1 \leq u \leq 1 \end{aligned}$$

The hint gives that $u = \text{sat}(-\lambda_1/(2x_2)) = \text{sat}(-\mu_1/(2t))$.

c. Now all that remains is to find μ_1. We must have

$$-c = \int_0^{t_f} u(t)\ \mathrm{d}t = \int_0^{t_f} \text{sat}\left(-1\frac{\mu_1}{2t}\right)\ \mathrm{d}t$$

The integral can be split into two parts. Since $c > 0$ it is clear that the control action must be negative. Thus $\mu_1 > 0$.

$$\int_0^{t_f} \text{sat}\left(-1\frac{\mu_1}{2t}\right)\ \mathrm{d}t = \int_0^{\mu_1/2} -1\ \mathrm{d}t + \int_{\mu_1/2}^{t_f} -1\frac{\mu_1}{2t}\ \mathrm{d}t$$

This is equal to

$$-\frac{\mu_1}{2} + \left[-\frac{\mu_1}{2}\log(t)\right]_{\mu_1/2}^{t_f} = -\frac{\mu_1}{2}\Big(-1 - \log(t_f) + \log(\mu_1/2)\Big)$$

μ_1 is thus given by the solution to

$$-c = -\frac{\mu_1}{2}\Big(-1 - \log(t_f) + \log(\mu_1/2)\Big)$$