Stability Definitions

An equilibrium point \( x = 0 \) of \( \dot{x} = f(x) \) is

- **locally stable**, if for every \( R > 0 \) there exists \( r > 0 \), such that
  \[ \|x(0)\| < r \Rightarrow \|x(t)\| < R, \ t \geq 0 \]

- **locally asymptotically stable**, if locally stable and
  \[ \|x(0)\| < r \Rightarrow \lim_{t \to \infty} x(t) = 0 \]

- **globally asymptotically stable**, if locally stable and
  \[ \lim_{t \to \infty} x(t) = 0 \]

for all \( x(0) \in \mathbb{R}^n \).

Lyapunov Theorem for Local Stability

**Theorem** Let \( \dot{x} = f(x), f(0) = 0, \) and \( 0 \in \Omega \subset \mathbb{R}^n \) for some open set \( \Omega \). Assume that \( V : \Omega \to \mathbb{R} \) is a \( C^1 \) function. If

- \( V(0) = 0 \)
- \( V(x) > 0, \) for all \( x \in \Omega, x \neq 0 \)
- \( \dot{V}(x) \leq 0 \) along all trajectories in \( \Omega \)

then \( x = 0 \) is locally stable. Furthermore, if also

- \( \dot{V}(x) < 0 \) for all \( x \in \Omega, x \neq 0 \)

then \( x = 0 \) is locally asymptotically stable.
Lyapunov Theorem for Global Stability

**Theorem** Let \( \dot{x} = f(x) \) and \( f(0) = 0 \). Assume that \( V : \mathbb{R}^n \to \mathbb{R} \) is a \( C^1 \) function. If

- \( V(0) = 0 \)
- \( V(x) > 0 \) for all \( x \neq 0 \)
- \( \dot{V}(x) < 0 \) for all \( x \neq 0 \)
- \( V(x) \to \infty \) as \( \|x\| \to \infty \) (radial unboundedness)

then \( x = 0 \) is **globally** asymptotically stable.

### Invariant Set Theorem

**Theorem** Let \( \Omega \in \mathbb{R}^n \) be a bounded and closed set that is invariant with respect to

\[ \dot{x} = f(x). \]

Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function such that \( \dot{V}(x) \leq 0 \) for \( x \in \Omega \). Let \( E \) be the set of points in \( \Omega \) where \( \dot{V}(x) = 0 \), if \( M \) is the largest invariant set in \( E \), then every solution with \( x(0) \in \Omega \) approaches \( M \) as \( t \to \infty \).

### Example

Example:
\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -x_2 - x_1^2 \]

Try with
\[ V(x) = x_2^2 + x_2^2 \] (Alt. 1)
or
\[ V(x) = 0.5x_1^4 + x_2^2 \] (Alt. 2)

### The Circle Criterion, \( 0 < k_1 \leq k_2 < \infty \)

**Theorem** Consider a feedback loop with \( y = Gu \) and \( u = -f(y) \). Assume \( G(s) \) is stable and that

\[ k_1 \leq \frac{f(y)}{y} \leq k_2. \]

If the Nyquist curve of \( G(s) \) stays outside the circle defined by the points \(-1/k_1 \) and \(-1/k_2\), then the closed-loop system is BIBO stable.

### Circle criterion / Sector conditions

What does it mean that we can get different sectors when using the circle criterion for a nonlinearity in feedback with a (fixed) linear system?

Can I have many different sector conditions, and what does that mean?

- **G**: stable system
  - \( 0 < k_1 < k_2 \): Stay outside circle
  - \( 0 = k_1 < k_2 \): Stay to the right of the line \( \text{Re } s = -1/k_2 \)
  - \( k_1 < 0 < k_2 \): Stay inside the circle

Other cases: Multiply \( f \) and \( G \) with \(-1\).
In the example above, the circle criterion can guarantee absolute stability for a nonlinearity which is bounded to either the sector \([k_1, k_2]\) or \([k_3, k_4]\) or in many other sectors, but NOT for a nonlinearity which is allowed to have a full variation within the sector \([k_1, k_4]\).

Example: \(G(s) = \frac{1000}{(s + 10)(s^2 + 2s + 100)}\) in negative feedback with a sector bounded nonlinearity.

Is it possible to draw phase portraits for systems of order higher than two?

Can the describing function method be improved by including more coefficients from the Fourier series expansion?

Are there criteria to verify the low-pass character needed in a describing function argument?

**Questions**

Example from exam 20090601 (a)

\[ e(t) = A \sin \omega t = \text{Im} (A e^{i\omega t}) \]

\[ u(t) = u_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \]

\[ u_1(t) = a_1 \cos \omega t + b_1 \sin \omega t = \text{Im} (N(A, \omega)A e^{i\omega t}) \]

where the describing function is defined as

\[ N(A, \omega) = \frac{b_1(\omega) + ia_1(\omega)}{A} \implies U(i\omega) \approx N(A, \omega)E(i\omega) \]

**Existence of Limit Cycles**

\[ y = G(\omega)u = -G(\omega)N(A)y \implies G(\omega) = -\frac{1}{N(A)} \]

The intersections of \(G(\omega)\) and \(-1/N(A)\) give \(\omega\) and \(A\) for possible limit cycles. Harder if \(N\) is a function of both \(A\) and \(\omega\).
Example from exam 20090601 (b)

Below we have the Nyquist and Bode curves of a stable linear system \( G \). Assume that there exists non-linearities corresponding to the three describing functions on previous page, and that each of these would be used in a negative feedback connection with \( G \). For which do we possibly get limit cycles? If so, state possible amplitudes of the limit cycles and if they are stable or unstable?

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Since the third describing function fulfills that \(- \frac{\omega_0}{\sqrt{2}} = -\frac{1}{2}\) and \( G(i\omega_0) \approx -0.6 \), we understand that we have two intersections. The first intersection occurs when \( A \approx 1.8 \) and the second intersection occurs when \( A \approx 4.5 \).

Examining the describing function around the first intersection, we see that \( \frac{1}{\sqrt{2}} \) goes from the outside of \( G(i\omega) \) to the inside, with increasing \( A \). Hence, we conclude that the possible limit cycle at \( A \approx 1.8 \) is unstable. By similar argument, we understand that the possible limit cycle at \( A \approx 4.5 \) is stable.

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Question

Please repeat the most important facts about sliding modes.

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Example

\[
\dot{x}_1 = 1 - u/4 \\
\dot{x}_2 = u, \quad u = -\text{sign } x_2, \quad \text{(i.e., } \sigma(x) = x_2) \tag{1}
\]

What is the sliding set and what is the sliding dynamics for the system above?

If \( \sigma(x) > 0 \Rightarrow u = -1 \Rightarrow f^+ = \begin{bmatrix} 5/4 \\ -1 \end{bmatrix} \)

\( \sigma(x) < 0 \Rightarrow u = +1 \Rightarrow f^- = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \)
The sliding dynamics:

Alternative 1: Solve via normal projection on σ:
Pick α such that for \( \dot{x} = \alpha f^+ + (1 - \alpha) f^- \), we get
\[ \dot{\sigma} = 0 \Rightarrow \dot{x}_2 = \alpha f^+_2 + (1 - \alpha) f^-_2 = 0 \]
This gives \( \alpha = 1/2 \), hence \( \dot{x} = \alpha f^+ + (1 - \alpha) f^- \) and \( \dot{x}_1 = 1 \) is the sliding dynamics.

Alternative 2: Solve via Equivalent control
\[ \dot{\sigma}(x)u = u_{eq} = 0 \] and \( \dot{\sigma} = \dot{x}_2 = u \Rightarrow u_{eq} = 0 \). Hence \( \dot{x}_1 = 1 - u_{eq}/4 = 1 \) is the sliding dynamics.

Problem Formulation (1)

Minimize \( \int_{t_0}^{t_f} L(x(t), u(t)) \, dt + \phi(x(t_f)) \)
\[ \dot{x}(t) = f(x(t), u(t)) \]
\[ u(t) \in U, \quad 0 \leq t \leq t_f, \quad t_f \text{ given} \]
\[ x(0) = x_0 \]
\[ x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m \]
\[ U \text{ control constraints} \]

Problem Formulation (2)

As in (1) but with additions:

- \( r \) end constraints
  \[ \Psi(x(t_f)) = \begin{bmatrix} \Psi_1(x(t_f)) \\ \vdots \\ \Psi_m(x(t_f)) \end{bmatrix} = 0 \]
- free end time \( t_f \)

Free end time \( t_f \)

If the choice of \( t_f \) is included in the optimization and/or final state constraints, then two cases: \( n_0 = 1 \) or \( n_0 = 0 \).
Also, if the choice of \( t_f \) is included in the optimization, there is an extra constraint:
\[ H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0 \]

The Maximum Principle (18.2)

Introduce the Hamiltonian
\[ H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u) \]
Suppose optimization problem (1) has a solution \( x^*(t), u^*(t) \). Then the optimal solution must satisfy
\[ \min_{u \in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad 0 \leq t \leq t_f \]
where \( \lambda(t) \) solves the adjoint equation
\[ \dot{\lambda}(t) = -H_x^T(x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_f) = \phi_x^T(x^*(t_f)) \]
where \( H_x = \frac{\partial H}{\partial x} \), \( \phi_x = \frac{\partial \phi}{\partial x} \).

The Maximum Principle–General Case (18.4)

Introduce the Hamiltonian
\[ H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda^T f(x, u) \]
Suppose optimization problem (2) has a solution \( x^*(t), u^*(t) \). Then there is a vector function \( \lambda(t) \), a number \( n_0 \geq 0 \), and a vector \( \mu \in \mathbb{R}^n \) so that \( [n_0 \mu]^T \neq 0 \) and
\[ \min_{u \in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad 0 \leq t \leq t_f \]
where
\[ \dot{\lambda}(t) = -H_x^T(x^*(t), u^*(t), \lambda(t), n_0) \]
\[ \lambda(t_f) = n_0 \phi_x^T(x^*(t_f)) \]

Example: Optimal storage control

Minimize \( \int_{t_0}^{t_f} [u(t) r^T + cx(t)] \, dt \)
subject to
\[ \begin{cases} \dot{x} = u, & 0 \leq u \leq M \\ x(0) = 0 \\ x(t_f) \geq A \end{cases} \]
\[ x = \text{ stock size} \]
\[ u = \text{ production rate} \]
\[ r = \text{ production cost growth rate} \]
\[ c = \text{ storage cost} \]

Question

Please repeat optimal control with some additional example
Extra session before the exam

Extra session for questions:

Thursday January 9 at 13.15-15.00 in the seminar room 2112B of
Automatic Control.

**Example: Optimal storage control I**

in standard form

\[
\begin{align*}
\text{Minimize} & \quad \int_0^{t_f} (c x_1(t) + u(t) x_2(t)) \, dt \\
\text{subject to} & \quad \dot{x}_1 = u, \quad \dot{x}_2 = x_1 x_2 \\
& \quad x_1(0) = 0, \quad x_2(0) = 1 \\
& \quad 0 \leq u \leq M \\
& \quad x_1(t_f) = A \\
L(u, x) & = u x_2 + c x_1 \quad \text{running cost} \\
\phi(x) & = x_1 \quad \text{final cost} \\
\psi(x) & = x_1 \quad \text{final constraint} \\
\end{align*}
\]

**Optimal storage control II**

Hamiltonian

\[
H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda(t)^T f(x, u) = n_0 (u x_2 + c x_1) + \lambda_1 u + \lambda_2 x_2
\]

Adjoint equations

\[
\begin{align*}
\lambda_1(t) & = -\frac{\partial H}{\partial x_1} = -n_0 c \quad \lambda_2(t) = -\frac{\partial H}{\partial x_2} = -n_0 u - \lambda_2 x_2 \\
\lambda_1(t_f) & = n_0 \frac{\partial \lambda_1}{\partial x_1}(x^*(t_f)) + \mu \frac{\partial \psi}{\partial x_1}(x^*(t_f)) = \mu \\
\lambda_2(t_f) & = n_0 \frac{\partial \lambda_2}{\partial x_2}(x^*(t_f)) + \mu \frac{\partial \psi}{\partial x_2}(x^*(t_f)) = 0
\end{align*}
\]

Should try two cases:

*normal case* $n_0 = 1$ and $\mu \geq 0$

*abnormal case* $n_0 = 0$ and $\mu > 0$

**Optimal storage control III**

Abnormal case: $n_0 = 0, \mu > 0$

\[
\lambda_1(t) = \mu \quad \forall 0 \leq t \leq t_f
\]

For every $0 \leq t \leq t_f$

\[
u^*(t) \in \text{argmin} \ H(x^*, u, \lambda, 0) = \text{argmin} \{\mu u\}
\]

\[
u^*(t) = 0 \quad \forall 0 \leq t \leq t_f
\]

violates constraint

\[
x_1(t_f) = A
\]

**Optimal storage control IV**

Normal case: $n_0 = 1, \mu \geq 0$

\[
\lambda_1(t) = b + ct, \quad b = \mu - ct_f \quad x_2(t) = e^{rt}
\]

For every $0 \leq t \leq t_f$

\[
u^*(t) \in \text{argmin} \ H(x^*, u, \lambda, 1) = \text{argmin} \{u(e^{rt} + b - ct)\}
\]

\[
u^*(t) = \begin{cases} M & \text{if } e^{rt} + b - ct < 0 \\
0 & \text{if } e^{rt} + b - ct > 0 \\
\end{cases}
\]

\[
u^*(t) = \begin{cases} M & \text{if } e^{rt} + b - ct < 0 \\
0 & \text{if } e^{rt} + b - ct > 0 \\
\end{cases} \quad t_1 \leq t \leq t_2
\]

\[
x(t_f) = A \quad \text{gives that} \quad M(t_f - t_1) = A. \quad \text{To find} \quad t_1, \quad \text{solve}
\]

\[
\min_{0 \leq s \leq A/M} \left\{ \int_s^{s+A/M} M(e^{rt} + ct) \, dt + \int_s^{t_f} cAdt \right\}
\]