Lecture 4 — Lyapunov Stability

Material
- Glad & Ljung Ch. 12.2
- Khalil Ch. 4.1-4.3
- Lecture notes

Today’s Goal
To be able to
- prove local and global stability of an equilibrium point using Lyapunov’s method
- show stability of a set (e.g., an equilibrium, or a limit cycle) using La Salle’s invariant set theorem.

Alexandr Mihailovich Lyapunov (1857–1918)

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Main idea
Lyapunov formalized the idea:
- If the total energy is dissipated, then the system must be stable.

Main benefit: By looking at how an energy-like function \( V \) (a so called Lyapunov function) changes over time, we might conclude that a system is stable or asymptotically stable without solving the nonlinear differential equation.

Analysis: Check if \( V \) is decreasing with time:
- Continuous time: \( \frac{dV}{dt} < 0 \)
- Discrete time: \( V(k+1) - V(k) < 0 \)

Main question: How to find a Lyapunov function?

Examples
Start with a Lyapunov candidate \( V \) to measure e.g.,
- “size” of state and/or output error,
- “size” of deviation from true parameters,
- energy difference from desired equilibrium,
- weighted combination of above
- ...

A Motivating Example
\[
\begin{align*}
\dot{x} &= -b|\dot{x}|\dot{x} - k_0 x - k_1 x^3 \\
\text{Total energy} &= \text{kinetic + pot. energy: } V(x, \dot{x}) = \frac{m\dot{x}^2}{2} + k_0 x^2/2 + k_1 x^4/4 > 0, \quad V(0,0) = 0 \\
\frac{d}{dt} V(x, \dot{x}) &= m\ddot{x} + k_0 \dot{x} - k_1 x^3 = \{\text{plug in system dynamics}\} \\
&= -b|\dot{x}|^3 < 0, \text{ for } \dot{x} \neq 0
\end{align*}
\]
What does this mean?

Stability Definitions
An equilibrium point \( x^\ast \) of \( \dot{x} = f(x) \) (i.e., \( f(x^\ast) = 0 \)) is
- locally stable, if for every \( R > 0 \) there exists \( r > 0 \), such that
  \( \|x(0) - x^\ast\| < r \implies \|x(t) - x^\ast\| < R, \quad t \geq 0 \)
- locally asymptotically stable, if locally stable and
  \( \|x(0) - x^\ast\| < r \implies \lim_{t \to \infty} x(t) = x^\ast \)
- globally asymptotically stable, if asymptotically stable for all \( x(0) \in \mathbb{R}^n \).

Lyapunov Theorem for Local Stability
Theorem
Let \( \dot{x} = f(x) \), \( f(x^\ast) = 0 \) where \( x^\ast \) is in the interior of \( \Omega \subset \mathbb{R}^n \). Assume that \( V : \Omega \to \mathbb{R} \) is a \( C^1 \) function. If

1. \( V(x^\ast) = 0 \)
2. \( V(x) > 0 \) for all \( x \in \Omega, x \neq x^\ast \)
3. \( \dot{V}(x) \leq 0 \) along all trajectories of the system in \( \Omega \)

then \( x^\ast \) is locally stable.

If also

4. \( \dot{V}(x) < 0 \) for all \( x \in \Omega, x \neq x^\ast \)

then \( x^\ast \) is locally asymptotically stable.

Furthermore, if \( \Omega = \mathbb{R}^n \) and also

5. \( V(x) \to \infty \text{ as } \|x\| \to \infty \)

then \( x^\ast \) is globally asymptotically stable.
Lyapunov Functions (≈ Energy Functions)

A function $V$ that fulfills (1)–(3) is called a Lyapunov function.

Condition (3) means that $V$ is non-increasing along all trajectories in $\Omega$:

$\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = \sum_i \frac{\partial V}{\partial x_i} f_i(x) \leq 0$

where $\frac{\partial V}{\partial x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \ldots & \frac{\partial V}{\partial x_n} \end{bmatrix}$

Geometric interpretation

Vector field points into sublevel sets

Trajectories can only go to lower values of $V(x)$

Lyapunov’s Linearization Method

Consider $\dot{x} = f(x)$

Assume that $f(0) = 0$.

$\dot{x} = Ax + g(x), \quad \|g(x)\| = o(\|x\|)$ as $x \to 0$.

Find Lyapunov function for linearization $\dot{x}$. Use it for original system (near the equilibrium)

Positive Definite Matrices

Definition: Symmetric matrix $M = M^T$ is

- positive definite $\langle M > 0$ if $x^T M x > 0, \forall x \neq 0$
- positive semidefinite $\langle M \geq 0$ if $x^T M x \geq 0, \forall x$

Lemma:

- $M = M^T > 0 \iff \lambda_i(M) > 0, \forall i$
- $M = M^T \geq 0 \iff \lambda_i(M) \geq 0, \forall i$

$M = M^T > 0 \quad V(x) := x^T M x$

$V(0) = 0, \quad V(x) > 0, \forall x \neq 0$

$V(x)$ candidate Lyapunov function

Lyapunov Stability for Linear Systems

Linear system: $\dot{x} = Ax$

Lyapunov equation: Let $Q = Q^T > 0$. Solve

$PA + A^T P = -Q$

with respect to the symmetric matrix $P$.

Lyapunov function: $V(x) = x^T P x$, $\Rightarrow$

$V(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P)x = -x^T Qx < 0$

Asymptotic Stability: If $P = P^T > 0$, then the Lyapunov Stability

Theorem implies (local–global) asymptotic stability, hence the eigenvalues of $A$ must satisfy $\text{Re} \lambda_k(A) < 0$, $\forall k$
**Converse Theorem for Linear Systems**

If Re \( \lambda_k(A) < 0 \) \( \forall k \), then for every \( Q = Q^T > 0 \) there exists \( P = P^T > 0 \) such that \( PA + A^TP = -Q \)

*Proof:* Choose \( P = \int_0^\infty e^{AT}Qe^{A^T}dt \). Then

\[
A^TP + PA = \lim_{t \to \infty} \int_0^t \left( A^T e^{AT}Qe^{A^T} + e^{AT}Qe^{A^T} \right) dt
= \lim_{t \to \infty} \left[ A^T Q e^{A^T} \right]_0^t
= -Q
\]

**Example: Lyapunov function for linear system**

\[
\dot{x} = Ax = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

(1)

Eigenvalues of \( A \): \((-1, -3) \Rightarrow \) (global) asymptotic stability.

Find a quadratic Lyapunov function for the system (1):

\[ V(x) = x^T P x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad P = P^T > 0 \]

Take any \( Q = Q^T > 0 \), say \( Q = I_{2 \times 2} \). Solve \( A^TP + PA = -Q \).

**Example cont’d**

\[
A^TP + PA = -I
\]

\[
\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 8p_{12} + 4p_{11} & 0 \\ 0 & -1 \end{bmatrix}
\]

Solving for \( p_{11}, p_{12} \) and \( p_{22} \) gives

\[
\begin{align*}
2p_{11} &= -1 \\
-4p_{12} + 4p_{11} &= 0 \\
8p_{12} - 6p_{22} &= -1
\end{align*}
\]

\[\Rightarrow \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 5/6 \end{bmatrix} > 0\]

Interpretation

Assume \( \dot{x} = Ax, x(0) = z \). Then

\[
\int_0^\infty x^T(t)Qx(t)dt = z^T \left( \int_0^\infty e^{AT}Qe^{A^T}dt \right) z = z^T P z
\]

Thus \( V(z) = z^T P z \) is the cost-to-go from \( z \) (with no input) and integral quadratic cost function with weighting matrix \( Q \).

**Proof of (1) in Lyapunov’s Linearization Method**

Put \( V(x) = x^T P x \). Then, \( V(0) = 0, V(x) > 0 \forall x \neq 0 \), and

\[\dot{V}(x) = x^T Pf(x) + f^T(x)P x = x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]P x = x^T (PA + A^TP)x + 2x^T P g(x) = -x^T Q x + 2x^T P g(x)\]

\[x^T Q x \geq \lambda_{\text{min}}(Q) \| x \|^2\]

and for all \( \gamma > 0 \) there exists \( r > 0 \) such that

\[\| g(x) \| < \gamma \| x \|, \quad \forall \| x \| < r\]

Thus, choosing \( \gamma \) sufficiently small gives

\[\dot{V}(x) \leq - (\lambda_{\text{min}}(Q) - 2\gamma \lambda_{\text{max}}(P)) \| x \|^2 < 0\]
Lyapunov Theorem for Global Asymptotic Stability

Theorem Let $\dot{x} = f(x)$ and $f(x^*) = 0$. If there exists a $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that

1. $V(x^*) = 0$
2. $V(x) > 0$, for all $x \neq x^*$
3. $\dot{V}(x) < 0$ for all $x \neq x^*$
4. $V(x) \to \infty$ as $\|x\| \to \infty$

then $x^*$ is a globally asymptotically stable equilibrium.

Radial Unboundedness is Necessary

If the condition $V(x) \to \infty$ as $\|x\| \to \infty$ is not fulfilled, then global stability cannot be guaranteed.

Example Assume $V(x) = x_1^2/(1 + x_1^2) + x_2^2$ is a Lyapunov function for a system. Can have $\|x\| \to \infty$ even if $\dot{V}(x) < 0$.

Somewhat Stronger Assumptions

Theorem: Let $\dot{x} = f(x)$ and $f(x^*) = 0$. If there exists a $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ and numbers $\alpha, \varepsilon, \sigma > 0$ such that

1. $\dot{V}(x) \leq -\alpha V(x)$ for all $x$
2. $V(x) \to \infty$ as $\|x\| \to \infty$

then $x^*$ is globally exponentially stable.

Proof Idea

Assume $x(t) \neq 0$ (otherwise we have $x(\tau) = 0$ for all $\tau > t$). Then

$$\frac{\dot{V}(x)}{V(x)} \leq -\alpha$$

Integrating from $0$ to $t$ gives

$$\log V(x(t)) - \log V(x(0)) \leq -\alpha t \Rightarrow V(x(t)) \leq e^{-\alpha t} V(x(0)).$$

Hence, $V(x(t)) \to 0$ exponentially as $t \to \infty$.

From (2) it follows that also $|x(t) - x^*|$ decays exponentially.

Invariant Sets

Definition: A set $M$ is called invariant if for the system

$$\dot{x} = f(x),$$

$x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.

LaSalle’s Invariant Set Theorem

Theorem Let $\Omega \subseteq \mathbb{R}^n$ compact invariant set for $\dot{x} = f(x)$. Let $V : \Omega \to \mathbb{R}$ be a $C^1$ function such that $\dot{V}(x) \leq 0$, $\forall x \in \Omega$, $E := \{x \in \Omega : \dot{V}(x) = 0\}$, $M := \text{largest invariant subset of } E \Rightarrow \forall x(0) \in \Omega, x(t) \text{ approaches } M \text{ as } t \to +\infty$.

A Motivating Example (revisited)

$$m\ddot{x} = -h|\dot{x}| - k_0 x - k_1 x^3$$

$$V(x, \dot{x}) = (2m\ddot{x}^2 + 2k_0 x^2 + k_1 x^4)/4 \geq 0, \quad V(0, 0) = 0$$

$$\dot{V}(x, \dot{x}) = -h|\dot{x}| \text{ gives } E = \{(x, \dot{x}) : \dot{x} = 0\}.$$  

$M$ consists of points that not only are in $E$ (have $\dot{x} = 0$), but also stay there (have $x = 0$). When $\dot{x} = 0$, we must also have $x = 0$.

Hence, $M = \{(0, 0)\}$ so LaSalle’s theorem gives global asymptotic stability of the origin.

Special Case: Global Stability of Equilibrium

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that

1. $V(0) = 0, V(x) > 0$ for all $x \neq 0$
2. $\dot{V}(x) \leq 0$ for all $x$
3. $V(x) \to \infty$ as $\|x\| \to \infty$
4. The only solution of $\dot{x} = f(x), V(x) = 0$ is $x(t) = 0 \forall t$ \Rightarrow $x = 0$ is globally asymptotically stable.

Radial Unboundedness is Necessary

If the condition $V(x) \to \infty$ as $\|x\| \to \infty$ is not fulfilled, then global stability cannot be guaranteed.
Example—Stable Limit Cycle

Show that $M = \{ x : \|x\| = 1 \}$ is a asymptotically stable limit cycle for (almost globally, except for starting at $x = 0$):

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2)
\end{align*}
\]

Let $V(x) = (x_1^2 + x_2^2 - 1)^2$.

\[
\frac{dV}{dt} = 2(x_1^2 + x_2^2 - 1)\frac{d}{dx}(x_1^2 + x_2^2 - 1) = -2(x_1^2 + x_2^2 - 1)^2(x_1^2 + x_2^2) \leq 0 \text{ for } x \in \Omega
\]

$\Omega = \{ 0 < \|x\| \leq R \}$ is invariant for $R = 1$.

Let us try the Lyapunov function $V = \frac{1}{2}(\tilde{x}^2 + \gamma_a \tilde{a}^2 + \gamma_b \tilde{b}^2)$

\[
\dot{V} = \tilde{x} \dot{\tilde{x}} + \gamma_a \tilde{a} \dot{\tilde{a}} + \gamma_b \tilde{b} \dot{\tilde{b}} = -a \tilde{x} \tilde{x} - \tilde{a} \tilde{a} + \tilde{b} \tilde{b} = -ax^2
\]

where the last equality follows if we choose

\[
\tilde{a} = -\tilde{a}, \quad \tilde{b} = -\tilde{b} = -1
\]

This proves that $\tilde{x} \to 0$.

(The parameters $\tilde{a}$ and $\tilde{b}$ do not necessarily converge: $u \equiv 0$.)

Demonstration if time permits

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Adaptive Noise Cancellation by Lyapunov Design

\[
\begin{align*}
\dot{x} + ax &= bu \\
\dot{\tilde{x}} + \tilde{a} \tilde{x} &= \tilde{b} u
\end{align*}
\]

Introduce $\tilde{x} = x - \tilde{x}$, $\tilde{a} = a - \tilde{a}$, $\tilde{b} = b - \tilde{b}$.

Want to design adaptation law so that $\tilde{x} \to 0$

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Results

![Estimation of parameters](image)

Results

![Estimation of parameters](image)

Next Lecture

- Stability analysis using input-output (frequency) methods