LUND INSTITUTE
OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - April 5, 2013 at 14–19

Points and grades

All answers must include a clear motivation. The total number of points is 28. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other. *Preliminary* grades:

3: 13 – 17.5 points

4: 18 – 23.5 points

5: 24 – 28 points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik". Pocket calculator.

Good Luck!

1. Find and classify all three equilibrium points of the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2 + x_1x_2 + x_2^2 \\ \dot{x}_2 &= -x_1 - x_1^2 - x_1x_2\end{aligned}$$

(4 p)

Solution

The equation system that should be solved is

$$\begin{aligned}0 &= -x_1 + 2x_2 + x_1x_2 + x_2^2 \\ 0 &= -x_1 - x_1^2 - x_1x_2\end{aligned}$$

The second equation gives

$$x_1(-x_1 - 1 - x_2) = 0 \Longleftrightarrow x_1 = 0 \text{ or } x_2 = -x_1 - 1$$

- *Case* $x_1 = 0$:
  The first equation now gives

  $$x_2(x_2 + 2) = 0 \Longleftrightarrow x_2 = 0 \text{ or } x_2 = -2$$

  That is, equilibria $(0, 0)$ and $(0, -2)$.
- *Case* $x_2 = -x_1 - 1$:
  Now, putting $x_2 = -x_1 - 1$ into the first equation and simplifying

  $$-2x_1 - 1 = 0 \Longleftrightarrow x_1 = -\frac{1}{2},\ x_2 = -\frac{1}{2}$$

  That is, equilibrium $(-\frac{1}{2}, -\frac{1}{2})$.

To classify each equilibrium, the Jacobian is determined

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + x_2 & 2 + x_1 + 2x_2 \\ -1 - 2x_1 - x_2 & -x_1 \end{bmatrix}$$

- *Case* $x^0 = (0, 0)$:

  $$\frac{\partial f}{\partial x}(x^0) = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}$$

  which has the characteristic polynomial $s^2 + s + 2$, hence 2 complex-valued eigenvalues with negative real part, i.e. **stable focus**.
- *Case* $x^0 = (0, -2)$:

  $$\frac{\partial f}{\partial x}(x^0) = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

  which has the characteristic polynomial $s^2 + 3s + 2$, hence 2 negative real-valued eigenvalues, i.e. **stable node**.

- *Case* $x^0 = (-\frac{1}{2}, -\frac{1}{2})$:

$$\frac{\partial f}{\partial x}(x^0) = \frac{1}{2}\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$$

which has the characteristic polynomial s^2+2s-4, hence 2 real-valued eigenvalues, 1 negative and 1 positive, i.e. **saddle point**.

2. Consider the control system

$$\ddot{x} - 2(\dot{x})^2 + x = u - 1$$

a. Write the system in state-space form. (1 p)

b. Suppose $u(t) \equiv 0$. Find all equilibria and determine if they are stable or asymptotically stable if possible. (2 p)

c. Show that Eq. (2) is satisfied by the periodic solution $x(t) = \cos(t)$, $u(t) = \cos(2t)$. Linearize the system around this solution. (2 p)

d. Design a state-feedback controller $u = u(x, \dot{x})$ for (2), such that the origin of the closed loop system is globally asymptotically stable. (1 p)

Solution

a. Introduce $x_1 = x$, $x_2 = \dot{x}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 2x_2^2 + u - 1 \end{aligned} \tag{1}$$

b. Let $\dot{x}_1 = \dot{x}_2 = 0 \Rightarrow (x_1, x_2) = (-1, 0)$ is the only equilibrium. The linearization around this point is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 4x_2 \end{bmatrix}_{(x_1^o, x_2^o) = (-1, 0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The characteristic equation for the linearized system is $s^2+1 = 0 \Rightarrow s = \pm i$. We can not conclude stability of the nonlinear system from this.

c.

$$x = \cos(t) \Rightarrow \dot{x} = -\sin(t) \Rightarrow \ddot{x} = -\cos(t)$$

By inserting this in the system dynamics and using *e.g.,* $u = \cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$ we get

$$\ddot{x} - 2(\dot{x})^2 + x = -\cos(t) - 2\sin^2(t) + \cos(t) = 2 + \cos^2(t) - 2 = u - 1$$

which shows that the trajectory is a solution.
The linearized system is thus

$$\begin{aligned} \delta\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 4x_2 \end{bmatrix}_{(x_1^o, x_2^o) = (\cos(t), -\sin(t))} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \\ &= \begin{bmatrix} 0 & 1 \\ -1 & -4\sin(t) \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \end{aligned} \tag{2}$$

where

$$\delta x = \begin{bmatrix} x_1(t) - \cos(t) \\ x_2(t) - (-\sin(t)) \end{bmatrix}, \quad \delta u = u(t) - \cos(2t)$$

d. The simplest way is to cancel the constant term and the nonlinearity with the control signal and introduce some linear feedback.

$$u = +1 - 2(\dot{x}_2)^2 - a\dot{x}, \quad a > 0 \Rightarrow \ddot{x} + a\dot{x} + x = 0$$

As the resulting system is linear and time invariant with poles in the left half plane for all $a > 0$ it is GAS.

3. A nonlinear system is given below.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_2 - x_2^3 - x_1 \end{aligned}$$

Show that the origin is globally asymptotically stable using the Lyapunov function candidate $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. (3 p)

Solution

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = -3x_2^2 - x_2^4 - x_1x_2 + x_1x_2 \le 0$$

We need to use LaSalle's theorem. The set E, where $\dot{V} = 0$, is the set of all points where $x_2 = 0$. To use LaSalle's theorem, we need to find M, the set of points that not only are in E, but also stay there. Hence M consists of points where not only $x_2 = 0$, but also $\dot{x}_2 = 0$. This means that $M = \{(0,0)\}$, so LaSalle's theorem gives that the origin is globally asymptotically stable.

4. A linear time-invariant system $G(s)$ is feedback interconnected with the nonlinear function $-bf(y)$ according to Figure 1.

$$G(s) = \frac{1}{(s+1)(s+2)}$$

and b is a positive constant. The nonlinear function $f(y) = \sin(y)$ is shown in Figure 2, and the Nyquist curve of $G(i\omega)$ in Figure 3.

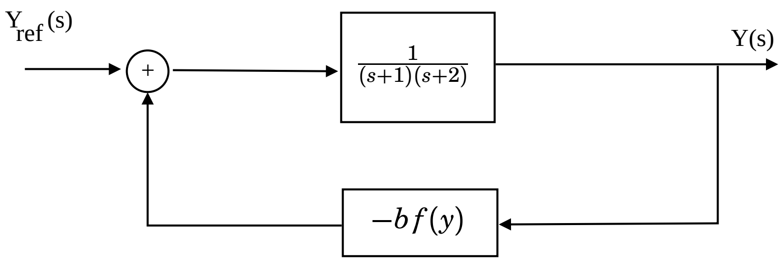

Figure 1 The block diagram for Problem 4(a)

Determine the largest value of b for which global asymptotic stability for the closed loop system is implied by:

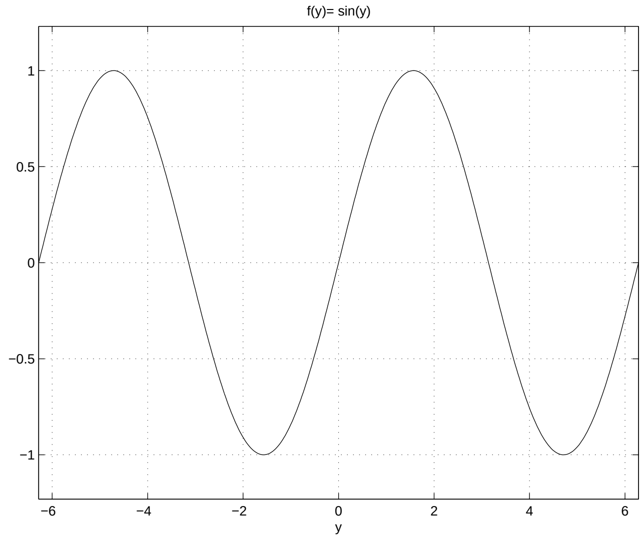

Figure 2 The nonlinear function $f(y) = \sin(y)$ in Problem 4 (a)

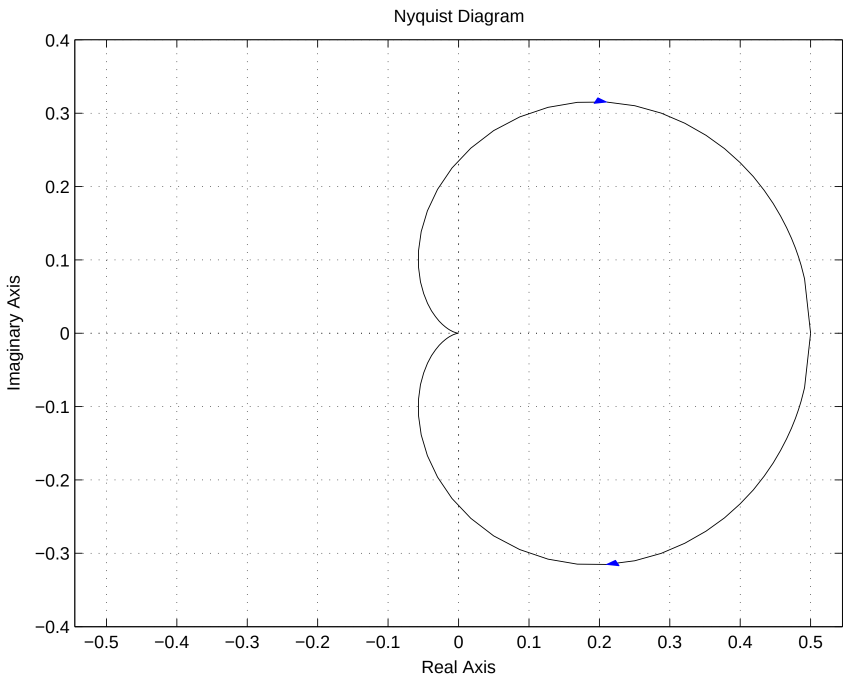

Figure 3 The Nyquist curve for Problem 4 (a)

a. The Small Gain Theorem (1.5 p)

b. the Circle Criterion (1.5 p)

Solution

a. The maximum gain of $G(s)$ can be determined from both the transfer function and the nyquist plot, and it is 0.5. The maximum gain of the nonlinearity is 1, given by the derivative of $f(y) = sin(y)$. Thus, it follows that $b < 2$ in order to guarantee stability using the Small Gain Theorem $(||f(y)||_\infty \cdot ||G(i\omega)||_\infty < 1)$.

b. The nonlinearity is contained within two sectors, given by graphically to $k_1 = -0.22$ and $k_2 = 1$. The circle criterion then guarantees stability if the nyquist curve is contained within the circle that passes through the points $-1/k_1$ and $-1/k_2$. It can be seen from the nyquist curve that the maximum possible radius of the circle is approx. 0.31, and since the circle radius is calculated as $r = (\frac{1}{k_2} - \frac{1}{k_1})/2b$, the maximum b is given by approx 9 (Fig. 4.

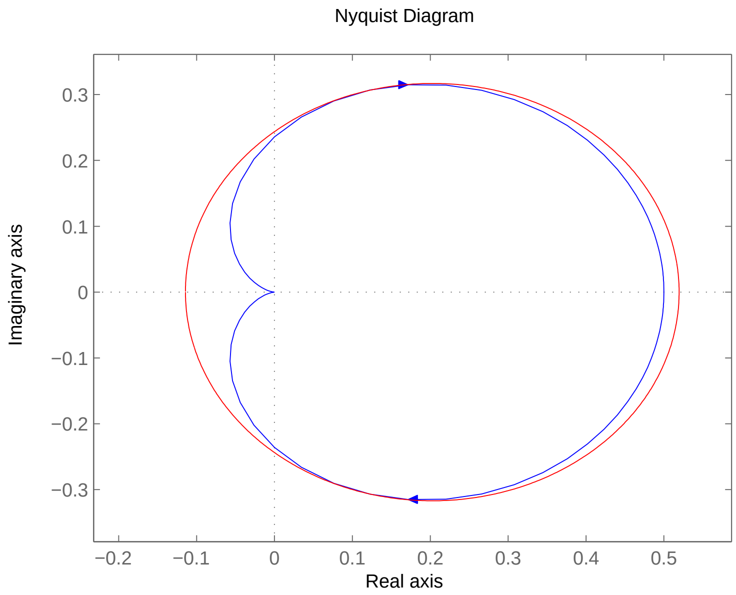

Figure 4 Circle criterion

5. Consider the system

$$\frac{\mathrm{d}^3 z}{\mathrm{d}t^3} + \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{1}{3}z^3$$

a. Show that the system can be written as a feedback connection as shown in Figure 5, where $P(s)$ is a transfer function and ψ is a static nonlinearity. (1 p)

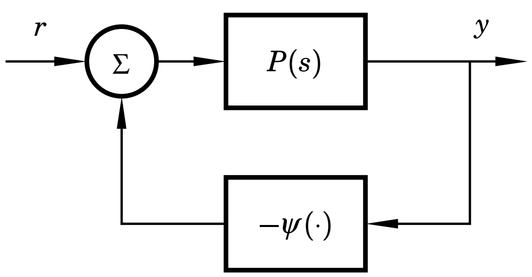

Figure 5 Figure for Problem 5

b. Calculate the describing function of the nonlinearity $\psi(x) = \frac{1}{3}x^3$. (2 p)
(*Hint:* $\int_0^{2\pi} \sin(x)^4 \mathrm{d}x = \frac{3\pi}{4}$)

c. Using the describing function method, analyze the existence, amplitude and frequency of possible limit cycles. (2 p)

Solution

a. Let $\psi = 1/3z^3$. Then a Laplace transform between ψ and z results in

$$P = \frac{1}{s(s^2+s+1)}.$$

The nonlinearity is $\psi = 1/3z^3$.

b. The function is odd, which implies that it is real.

$$b_1 = \frac{A^3}{3\pi}\int_0^{2\pi} \sin(\phi)^4 \mathrm{d}\phi = \frac{A^3}{4},$$

which gives that the describing function

$$N(A) = \frac{A^2}{4}.$$

c. We want to find out the points where $\mathrm{Im}P(i\omega) = 0$. Some calculations gives that

$$\mathrm{Im}P(i\omega) = \frac{-(1-\omega^2)}{\omega((1-\omega^2)^2+\omega^2)},$$

which in its turn gives that $\omega = 1$. Finally, this yields that

$$P(i) = -1 = -\frac{1}{N(A)} = -\frac{4}{A^2} \Rightarrow A = 2.$$

To conclude: The frequency of the limit cycle is $\omega = 1$ rad/s and its amplitude is $A = 2$.

6. Consider the system below:

$$\begin{aligned} \dot{x} &= -3x + u - \phi(x) \\ y &= x \end{aligned}$$

where ϕ is given by:

$$\phi(z) = z^5$$

Is the system BIBO stable from u to y? Hint: Try proving passivity or using the Circle Criterion. (3 p)

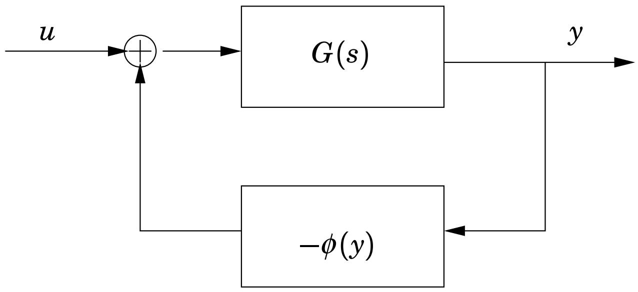

Figure 6 A linear system in feedback with a nonlinearity

Solution

We start by decomposing the system into one linear system in negative feedback with the nonlinaerity ϕ. See Figure 6.

ALT1. Circle criterion

ALT2. The transfer function G is given by:

$$G(s) = \frac{1}{s+3}$$

which is strictly passive since

$$\mathrm{Re}G(j\omega - \epsilon) = \mathrm{Re}\frac{1}{j\omega - \epsilon + 3} = \cdots = \frac{3-\epsilon}{\omega^2 + (3-\epsilon)^2} > 0$$

$\forall \omega > 0$, and e.g. $\epsilon = 1$. ϕ satisfies

$$z\phi(z) = z^6 \geq 0, \quad \forall z$$

and is therefore passive. BIBO stability then follows from the passivity theorem.

Note that the small gain theorem is not applicible in this case since gain of ϕ is not bounded.

7. A body under influence of a force obeys the equation

$$m\ddot{x} = F, \quad F_{\min} \leq F \leq F_{\max}.$$

Assume for simplicity that $m = 1$, $F_{\min} = -1$, $F_{\max} = 1$, and put $F = u$. Use Pontryagin's Maximum Principle to determine the optimal control $u(t)$ which allows the body to reach a rest in the origin in the shortest possible time, when starting from an arbitrary state $(x(0), \dot{x}(0))$. Specify whether the problem is normal or abnormal. (4 p)

Solution

The equations of motion are

$$\begin{aligned} \dot{x}_1 &= x_2, \quad x_1(0) = x_0, \quad x_1(T) = 0, \\ \dot{x}_2 &= u, \quad x_2(0) = v_0, \quad x_2(T) = 0, \\ u &\in [-1, 1]. \end{aligned}$$

The problem to solve is

$$\min \int_0^T 1 \mathrm{d}t.$$

The Hamilton function for the normal case ($n_o = 1$) is

$$H = 1 + \lambda_1 x_2 + \lambda_2 u,$$

which implies that the adjoint equations are

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0, \Rightarrow \lambda_1 = \lambda_1^0, \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1, \Rightarrow \lambda_2 = \lambda_2^0 - \lambda_1^0 t.
\end{aligned}$$

From this it follows that the control signal only changes sign at most once. Depending on the initial conditions, the expression for the optimal control trajectory is then either

$$u(t) = \begin{cases} 1 & , \quad 0 \le t \le t_1 \\ -1 & , \quad t_1 < t \le t_f \end{cases}$$

or

$$u(t) = \begin{cases} -1 & , \quad 0 \le t \le t_1 \\ 1 & , \quad t_1 < t \le t_f \end{cases}$$

where t_1 is the switching time and t_f is the final time.
The solution to the state equations stated earlier is given by

$$x(t_f) = e^{A_c t_f} x(0) + \int_0^{t_f} e^{A_c(t_f - \tau)} B_c u(\tau) d\tau \tag{3}$$

where for this problem

$$A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, e^{A_c t} = \begin{pmatrix} 1 & t \\ 0 & 1. \end{pmatrix},$$

For the case of positive control signal first, simplify the state equation solution:

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} + \int_0^{t_1} \begin{pmatrix} t_f - \tau \\ 1 \end{pmatrix} d\tau - \int_{t_1}^{t_f} \begin{pmatrix} t_f - \tau \\ 1 \end{pmatrix} d\tau \\
&= \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} + \begin{pmatrix} 2t_f t_1 - t_1^2 - t_f^2/2 \\ 2t_1 - t_f\,. \end{pmatrix}
\end{aligned} \tag{4}$$

The system above admits a solution (t_1, t_f) with $0 < t_1 < t_f$ if and only if

$$\frac{v_0^2}{2} - x_0 > 0\,, \qquad v_0 + \sqrt{\frac{v_0^2}{2} - x_0} > 0\,,$$

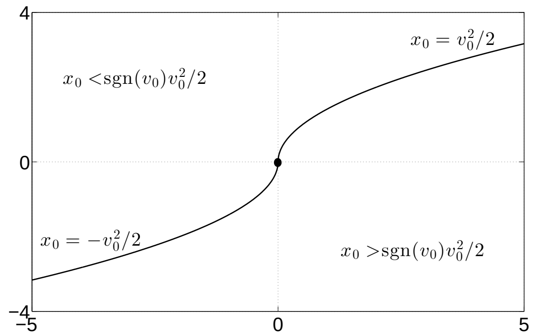

Figure 7 Phase plane for problem 7.

that is, if and only if

$$x_0 < \text{sign}(v_0)\frac{v_0^2}{2}.$$

If the above is satisfied, equation (4) is solved by

$$t_1 = \sqrt{\frac{v_0^2}{2} - x_0}, \qquad t_f = 2t_1 + v_0.$$

Similar calculations for the second case where $u(t)$ starts as -1 give

$$t_1 = \sqrt{\frac{v_0^2}{2} + x_0}, \qquad t_f = 2t_1 - v_0,$$

provided that $\frac{v_0^2}{2} + x_0 > 0$ and $t_f > t_1$, that is

$$x_0 > \text{sign}(v_0)\frac{v_0^2}{2}.$$

The case when $v_0 > 0$ and $x_0 = v_0^2/2$ can be treated as the first one with $t_0 = 0$ and $t_f = v_0$, that is, with the constant control $u(t) \equiv -1$. Symmetrically, the case $v_0 < 0$ and $x_0 = -v_0^2/2$ can be treated as the second one with $t_0 = 0$ and $t_f = -v_0$, that is, with the constant control $u(t) \equiv 1$. If $x_0 = v_0 = 0$, clearly $t_f = 0$ (we are starting already at rest in 0). See Fig.**??** for a plot of the different regions in the (x_0, v_0)-plane.

The answer is thus

$$u(t) = \begin{cases} 1 & , \quad 0 \le t \le \sqrt{\frac{v_0^2}{2} - x_0} \\ -1 & , \quad \sqrt{\frac{v_0^2}{2} - x_0} < t \le 2\sqrt{\frac{v_0^2}{2} - x_0} + v_0 \end{cases} \qquad \text{if } x_0 < \text{sgn}(v_0)v_0^2/2$$

$$u(t) = \begin{cases} -1 & , \quad 0 \le t \le \sqrt{\frac{v_0^2}{2} + x_0} \\ 1 & , \quad \sqrt{\frac{v_0^2}{2} + x_0} < t \le 2\sqrt{\frac{v_0^2}{2} + x_0} - v_0 \end{cases} \qquad \text{if } x_0 > \text{sgn}(v_0)v_0^2/2$$

$$u(t) = -1 \qquad 0 \le t \le v_0 \qquad \text{if } x_0 = v_0^2/2,\ v_0 > 0$$

$$u(t) = 1 \qquad 0 \le t \le -v_0 \qquad \text{if } x_0 = -v_0^2/2,\ v_0 < 0.$$

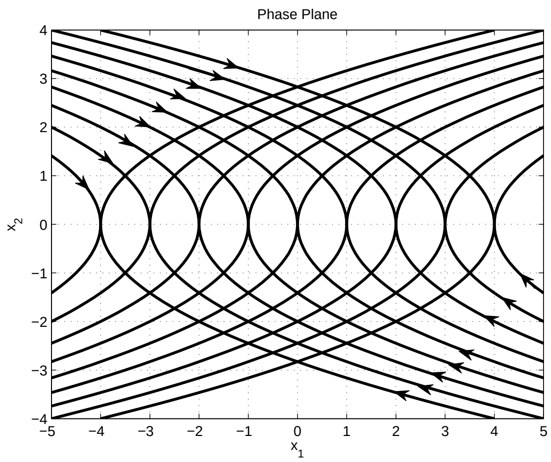

Figure 8 Phase plane for problem 7.

Graphical interpretation

Using

$$\frac{\mathrm{d}x_1}{\mathrm{d}x_2} = \frac{x_2}{u}, \Rightarrow x_1 = \frac{x_2^2}{2u} + C$$

the switching curve can be decided ($C = 0$ since the desired endpoint is the origin), and is given by

$$x_1 + \mathrm{sign}(x_2)(x_2^2/2) = 0, \tag{5}$$

This implies that the control signal can be written as

$$u(t) = -\mathrm{sign}(x_1 + \mathrm{sign}(x_2)(x_2^2/2)).$$

A phase plane is shown in Figure 7.

Since a solution to the normal case was found, the problem can be concluded to be normal.