Lecture 14 — Course Summary

- CEQ
- The exam
- Questions / review of the course

Exam (January 3, 2018, 8:00-13:00)

Course Material Allowed:
- Lecture slides 1-14 (no exercises or old exams)
- Laboratory exercises 1, 2, and 3
- Reglerteori by Glad and Ljung
- Applied Nonlinear Control by Slotine and Li
- Nonlinear Systems by Khalil
- Calculus of variations and optimal control theory by Liberzon

You may bring everything on the list + “Collection of Formulae for Control” to the exam.

Question: What’s on the exam?

Among old exam problems:
- Models, equilibria etc
- Linearization and stability
- Circle criterion
- Small gain
- Describing Functions
- Lyapunov functions
- Optimal Control
- ...

Old exams and solutions are available from the course home page.

Question

Can I get different answers if use the Small Gain theorem and the Circle criterion? What does it mean?
- If the conditions for stability are not satisfied for one criterion it does not necessarily mean that the system is unstable. It just means that you can not use that method to guarantee stability. You can never ‘prove’ that a system is stable with one method and ‘unstable’ with another.
- Similarly, there are no general guaranteed methods to find a Lyapunov function (though some suggested good methods/candidates are worth trying, e.g., quadratic, total energy, etc.).

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

- **locally stable**, if for every $R > 0$ there exists $r > 0$, such that $\|x(0)\| < r \Rightarrow \|x(t)\| < R$, $t \geq 0$
- **locally asymptotically stable**, if locally stable and $\|x(0)\| < r \Rightarrow \lim_{t \to \infty} x(t) = 0$
- **globally asymptotically stable**, if asymptotically stable for all $x(0) \in \mathbb{R}^n$.

Lyapunov Theorem for Local Stability

**Theorem** Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbb{R}^n$ for some open set $\Omega$. Assume that $V: \Omega \to \mathbb{R}$ is a $C^1$ function. If
- $V(0) = 0$
- $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- $\dot{V}(x) \leq 0$ along all trajectories in $\Omega$
then $x = 0$ is locally stable. Furthermore, if also
- $\dot{V}(x) < 0$, for all $x \in \Omega$, $x \neq 0$
then $x = 0$ is locally asymptotically stable.
Lyapunov Theorem for Global Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$. Assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ function. If
- $V(0) = 0$
- $V(x) > 0$, for all $x \neq 0$
- $V(x) < 0$, for all $x \neq 0$
- $\dot{V}(x) \to \infty$ as $\|x\| \to \infty$ (radial unboundedness)

then $x = 0$ is **globally** asymptotically stable.

Invariance Sets

Definition A set $M$ is called invariant if for the system $\dot{x} = f(x)$,

$x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.

Example

Example:

$\dot{x}_1 = x_2$  
$\dot{x}_2 = -x_2 - x_3^1$

Try with

$V(x) = x_2^2$ (Alt. 1)

or

$V(x) = 0.5x_1^4 + x_2^2$ (Alt. 2)

The Circle Criterion, $0 < k_1 \leq k_2 < \infty$

Theorem Consider a feedback loop with $y = f(y)$ and $u = G(u)$ Assume $G(s)$ is stable and that

$k_1 \leq \frac{f(y)}{y} \leq k_2$

If the Nyquist curve of $G(s)$ stays outside the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable.

Circle criterion / Sector conditions

What does it mean that we can get different sectors when using the circle criterion for a nonlinearity in feedback with a (fixed) linear system?

Can I have many different sector conditions, and what does that mean?

$G$: stable system

- $0 < k_1 < k_2$: Stay outside circle
- $0 = k_1 < k_2$: Stay to the right of the line $\text{Re } s = -1/k_2$
- $k_1 < 0 < k_2$: Stay inside the circle

Other cases: Multiply $f$ and $G$ with $-1$. 

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Circle criterion / Sector conditions

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Can I have many different sector conditions, and what does that mean?
Describing functions

In the example above, the circle criterion can guarantee absolute stability for a nonlinearity which is bounded to either the sector \([k_1, k_2]\) or \([k_3, k_4]\) or in many other sectors, but NOT for a nonlinearity which is allowed to have a full variation within the sector \([k_1, k_4]\).

Example: \(G(s) = \frac{1000}{(s + 10)(s^2 + 2s + 100)}\) in negative feedback with a sector bounded nonlinearity.

Questions

Is it possible to draw phase portraits for systems of order higher than two?
Can the describing function method be improved by including more coefficients from the Fourier series expansion?
Are there criteria to verify the low-pass character needed in a describing function argument?

Idea of Describing Function

Find frequency \(\omega\) and amplitude \(A\) such that

\(G(j\omega) \cdot N(A) = -1\)

Existence of Limit Cycles

\(y = G(i\omega)u = -G(i\omega)N(A)y \Rightarrow G(i\omega) = -\frac{1}{N(A)}\)

The intersections of \(G(i\omega)\) and \(-1/N(A)\) give \(\omega\) and \(A\) for possible limit cycles.

Harder if \(N\) is a function of both \(A\) and \(\omega\).

Example from exam 20090601 (a)

Which one of the three describing functions below corresponds to the nonlinearity \(f(x)\) above?

Figure: Describing functions 1 to 3
Example from exam 20090601 (b)

Below we have the Nyquist and Bode curves of a stable linear system $G$. Assume that there exists non-linearities corresponding to the three describing functions on previous page, and that each of these would be used in a negative feedback connection with $G$. For which do we possibly get limit cycles? If so, state possible amplitudes of the limit cycles and if they are stable or unstable?

Since the third describing function fulfills that $-\frac{1}{N(2)} = -\frac{1}{2}$ and $G(i\omega_0) \approx -0.6$, we understand that we have two intersections. The first intersection occurs when $A \approx 1.8$ and the second intersection occurs when $A \approx 4.5$.

Examining the describing function around the first intersection, we see that $-\frac{1}{N(2)}$ goes from the outside of $G(i\omega)$ to the inside, with increasing $A$. Hence, we conclude that the possible limit cycle at $A \approx 1.8$ is unstable. By similar argument, we understand that the possible limit cycle at $A \approx 4.5$ is stable.

What would the corresponding frequency of the limit cycles in (b) be?

The frequency of all possible limit cycles is approximately 2.5 rad/s. To understand this, we see in the Bode plot that for $\omega \approx 2.5$ we have that $\arg(G(i\omega)) \approx -180$.

Question

Please repeat the most important facts about sliding modes.

Sliding Modes

$\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}$

The sliding set is where $\sigma(x) = 0$ and $f^+$ and $f^-$ point towards $\sigma(x) = 0$.

The sliding dynamics are $\dot{x} = \alpha f^+ + (1 - \alpha)f^-$, where $\alpha$ is obtained from $\sigma = \frac{\partial f}{\partial x} = 0$.

(More precisely, find $\alpha$ such that the components of $f^+$ and $f^-$ perpendicular to the switching surface cancel.)

Example

$x_1 = 1 - u/4$
$x_2 = u, \quad (1)$

What is the sliding set and what is the sliding dynamics for the system above?

If

$\sigma(x) > 0 \Rightarrow u = -1 \Rightarrow f^+ = \begin{bmatrix} 5/4 \\ -1 \end{bmatrix}$

$\sigma(x) < 0 \Rightarrow u = +1 \Rightarrow f^- = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$

Sliding Mode Dynamics

The dynamics along the sliding surface $\sigma(x) = 0$ can also be obtained by setting $u = u_{eq} \in [-1, 1]$ such that $\sigma(x) = 0$. $u_{eq}$ is called the equivalent control.

The sliding set:

Find those values of the states at the switching curve for which

$\nabla \sigma \cdot f^+ < 0$

and

$\nabla \sigma \cdot f^- > 0$

(meaning that the vector fields on either side of $\sigma(x)$ points towards $\sigma(x)$, i.e., the normal projection of $f^+$ is negative and the normal projection of $f^-$ on $\sigma(x)$ is positive). If these conditions are not fulfilled we will just “flow through $\sigma(x)$.”

In this example all the values along $x_2 = 0$ will belong to the switching set. (Compare with example from lecture 9 where the switching set will be restricted to $x_2 = 0$ and $-1 \leq x_1 \leq 1$, see figure on slide above).
Alternative 1.: Solve via normal projection on $\sigma$:

Pick $\alpha$ such that for $\dot{x} = \alpha f^+ + (1 - \alpha) f^-$, we get

$\sigma = 0 \Rightarrow x_2 = \alpha f_2^+ + (1 - \alpha) f_2^- = 0$

This gives $\alpha = 1/2$, hence $\dot{x} = \alpha f^+ + (1 - \alpha) f^-$ and $x_1 = 1$ is the sliding dynamics.

Alternative 2: Solve via Equivalent control

$\sigma(x)_{x=x_0} = 0$ and $\dot{x} = u \Rightarrow u_{eq} = 0$.

Hence $x_1 = 1 - u_{eq}/4 = 1$ is the sliding dynamics.

Problem Formulation (1)

Minimize $\int_0^{t_f} L(x(t), u(t)) dt + \phi(x(t_f))$ subject to

$\dot{x}(t) = f(x(t), u(t))$

$u(t) \in U, 0 \leq t \leq t_f, t_f$ given

$x(0) = x_0$

$x(t) \in R^n, u(t) \in R^m$

$U$ control constraints

$\lambda(t) = \Psi(x(t), u(t))$

$\lambda(t_f) = \phi(x(t_f))$

Problem Formulation (2)

As in (1) but with additions:

- $r$ end constraints

  $\Psi(x(t_f)) = \begin{pmatrix} \Psi_1(x(t_f)) \\ \vdots \\ \Psi_r(x(t_f)) \end{pmatrix} = 0$

- free end time $t_f$

Free end time $t_f$

If the choice of $t_f$ is included in the optimization and/or final state constraints, then two cases: $n_0 = 1$ or $n_0 = 0$.

Also, if the choice of $t_f$ is included in the optimization, there is an extra constraint:

$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$

Example: Optimal storage control

Minimize $\int_0^{t_f} [u(t) e^t + c x(t)] dt$

subject to

$\begin{cases} x = u & 0 \leq t \leq M \\ x(0) = 0 \\ x(t_f) \geq A \end{cases}$

$x =$ stock size

$u =$ production rate

$r =$ production cost growth rate

$c =$ storage cost

The Maximum Principle (18.2)

Introduce the Hamiltonian

$H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u)$

Suppose optimization problem (1) has a solution $u^*(t), x^*(t)$. Then the optimal solution must satisfy

$\min_{u \in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), 0 \leq t \leq t_f$

where $\lambda(t)$ solves the adjoint equation

$\dot{\lambda}(t) = -H^*_x(x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_f) = \phi^*_x(x^*(t_f))$

where $H_x = \frac{\partial H}{\partial x} = [\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}], \phi_x = \frac{\partial \phi}{\partial x}$.

The Maximum Principle—General Case (18.4)

Introduce the Hamiltonian

$H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda^T f(x, u)$

Suppose optimization problem (2) has a solution $u^*(t), x^*(t)$. Then there is a vector function $\lambda^*(t)$, a number $n_0 \geq 0$, and a vector $\mu \in R^r$ so that $[n_0 \mu^T] \neq 0$ and

$\min_{u \in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), 0 \leq t \leq t_f$

where

$\dot{\lambda}(t) = -H^*_x(x^*(t), u^*(t), \lambda(t), n_0)$

$\lambda(t_f) = n_0 \phi^*_x(x^*(t_f)) + H^*_x(x^*(t_f)) \mu$

Question

Please repeat optimal control with some additional example
Example: Optimal storage control I

in standard form

\[
\begin{align*}
\text{Minimize} & \quad \int_0^{t_f} [cx_1(t) + u(t)x_2(t)]dt \\
\text{subject to} & \quad s_1 = u, \quad s_2 = rx_2 \\
& \quad x_1(0) = 0, \quad x_2(0) = 1 \\
& \quad 0 \leq u \leq M \\
& \quad x_1(t_f) = A \\
\end{align*}
\]

\[L(u, x) = ux_2 + cx_1 \text{ running cost} \]
\[\phi(x) = x_1 \text{ final cost} \]
\[\psi(x) = x_1 \text{ final constraint} \]
\[t_f \text{ fixed} \]

Optimal storage control III

Abnormal case: \( n_0 = 0, \) \( \mu > 0 \)
\[\lambda_1(t) = \mu \quad \forall 0 \leq t \leq t_f \]
For every \( 0 \leq t \leq t_f \)
\[u^*(t) \in \arg\min_u H(x^*, u, \lambda, 0) = \arg\min_u \{\mu u\} \]
\[u^*(t) = 0 \quad \forall 0 \leq t \leq t_f \]
violates constraint \( x_1(t_f) = A \)

Exercise sessions and before the exam

- No lectures next week, only exercises
- In addition questions can be asked on www.piazza.com.

Optimal storage control II

Hamiltonian
\[H(x, u, \lambda, n_0) = n_0L(x, u) + \lambda(t)^T f(x, u)\]
\[= n_0(ux_2 + cx_1) + \lambda_1u + \lambda_2rx_2\]

Adjoint equations
\[\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -n_0c \quad \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -n_0\mu - \lambda_2r\]
\[\lambda_1(t_f) = n_0 \frac{\partial}{\partial x_1} (x^*(t_f)) + \mu \frac{\partial}{\partial x_1} (x^*(t_f)) = \mu \]
\[\lambda_2(t_f) = n_0 \frac{\partial}{\partial x_2} (x^*(t_f)) + \mu \frac{\partial}{\partial x_2} (x^*(t_f)) = 0\]

Should try two cases:
- normal \( n_0 = 1 \) and \( \mu \geq 0 \)
- abnormal \( n_0 = 0 \) and \( \mu > 0 \)

Optimal storage control IV

Normal case: \( n_0 = 1, \mu \geq 0 \)
\[\lambda_1(t) = b - ct, \quad \lambda_2(t) = b \quad \lambda_1(t) = e^{rt}\]
For every \( 0 \leq t \leq t_f \)
\[u^*(t) \in \arg\min_u H(x^*, u, \lambda, 1) = \arg\min_u \{u(e^{rt} + b - ct)\} \]
\[u^*(t) = \begin{cases} 
M & \text{if } e^{rt} + b - ct < 0 \\
0 & \text{if } e^{rt} + b - ct > 0 
\end{cases} \]
\[u^*(t) = \begin{cases} 
M & \text{if } e^{rt} + b - ct < 0 \\
0 & \text{if } e^{rt} + b - ct > 0 
\end{cases} \]
\[x(t_f) = A \text{ gives that } M(t_2 - t_1) = A. \text{ To find } t_1, \text{ solve} \]
\[
\min_{0 \leq s \leq A/M} \left\{ \int_s^{s+A/M} M(e^{rt} + ct)dt + \int_{s+A/M}^{t_f} cAdt \right\}
\]