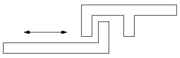
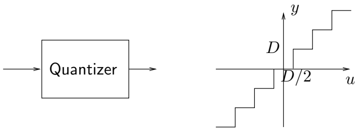
## Lecture 8 — Backlash and Quantization

**Today's Goal:**

- To know models and compensation methods for backlash



- Be able to analyze the effect of quantization errors

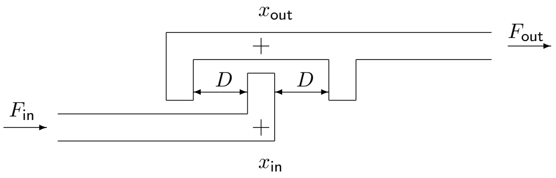


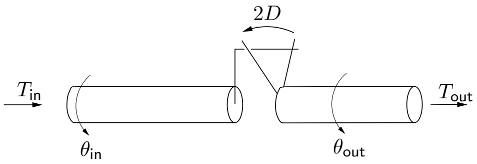
## Material

- Lecture slides

Note: We are using analysis methods from previous lectures (describing functions, small gain theorem etc.), and these have references to the course book(s).

## Linear and Angular Backlash





## Example: Parallel Kinematic Robot

Gantry-Tau robot: Need backlash-free gearboxes for high precision

EU-project: SMErobot$^{\text{TM}}$ `www.smerobot.org`

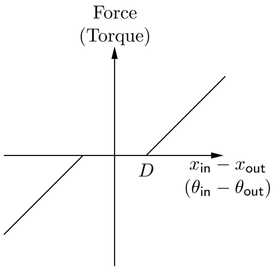
## Backlash

Backlash (*glapp*) is

- present in most mechanical and hydraulic systems
- increasing with wear
- bad for control performance
- may cause oscillations

Note: A gear box without any backlash will not work if temperature rises

## Dead-zone Model

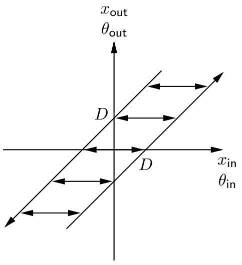


- Often easier to use model of the form $x_{\text{in}}(\cdot) \to x_{\text{out}}(\cdot)$
- Uses implicit assumption: $F_{\text{out}} = F_{\text{in}}, T_{\text{out}} = T_{\text{in}}$. Can be **wrong**, especially when "no contact".

## The Standard Model

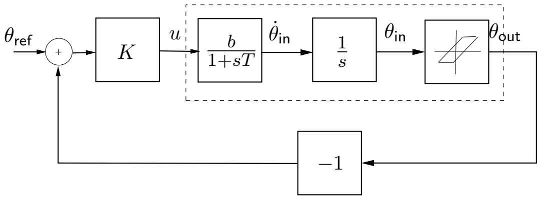
Assume instead

- $\dot{x}_{\text{out}} = \dot{x}_{\text{in}}$ when "in contact"
- $\dot{x}_{\text{out}} = 0$ when "no contact"
- No model of forces or torques needed/used



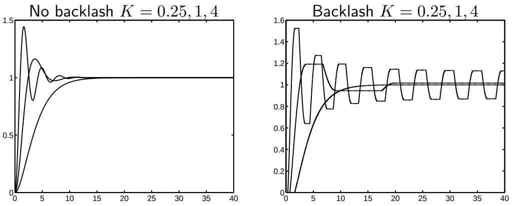
## Servo motor with Backlash

P-control of servo motor



How does the values of $K$ and $D$ affect the behavior?

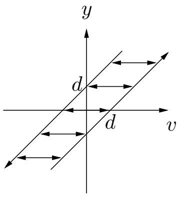
## Effects of Backlash

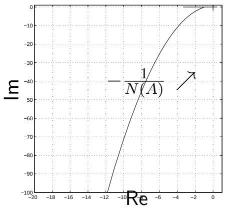


Oscillations for $K = 4$ but not for $K = 0.25$ or $K = 1$. Why?

Limit cycle becomes smaller if $D$ is made smaller, but it never disappears

## Describing Function for a Backlash





If $A > d$ then

$$N(A) = \frac{b_1 + ia_1}{A} \quad \text{with} \quad a_1 = \frac{4d}{\pi}\left(\frac{d}{A} - 1\right) \quad \text{and}$$

$$b_1 = \frac{A}{\pi}\left(\frac{\pi}{2} - \arcsin\left(\frac{2d}{A} - 1\right) + 2\left(1 - \frac{2d}{A}\right)\sqrt{\frac{d}{A}}\sqrt{1 - \frac{d}{A}}\right)$$

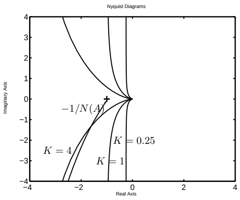
else $N(A) = 0$.

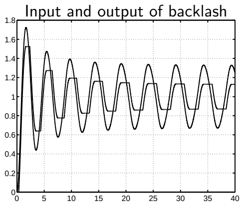
## 1 minute exercise

Study the plot for the describing function for the backlash on the previous slide.

Where does the function $-\dfrac{1}{N(A)}$ end for $A \to \infty$ and why?

## Describing Function Analysis



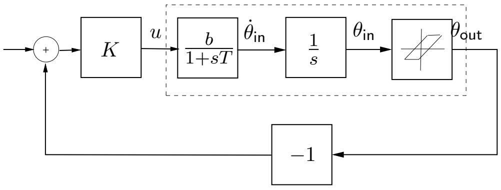


- For $K = 4, D = 0.2$: intersection between $G(j\omega)$ and $-1/N(A)$ occurs for $A = 0.33, \omega = 1.24$
- Simulation: $A = 0.33$, $\omega = 2\pi/5.0 = 1.26$
  Describing function predicts oscillation well!

## Limit cycles?

The describing function method is only approximate.

Can one determine conditions that **guarantee** stability?

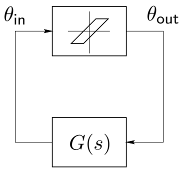


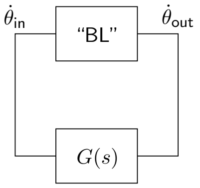
Note: $\theta_{\text{in}}$ and $\theta_{\text{out}}$ will not converge to zero

Idea: Consider instead $\dot{\theta}_{\text{in}}$ and $\dot{\theta}_{\text{out}}$

## Backlash Limit Cycles

Rewrite the system as





Note that the block "BL" satisfies

$$\dot{\theta}_{\text{out}} = \begin{cases} \dot{\theta}_{\text{in}} & \text{in contact} \\ 0 & \text{otherwise} \end{cases}$$

## Analysis by small gain theorem

Backlash block has gain $\le 1$ (from $\dot{\theta}_{\text{in}}$ to $\dot{\theta}_{\text{out}}$)

Hence closed loop is BIBO stable provided that

$G(s)$ is asymptotically stable and $|G(i\omega)| < 1$ for all $\omega$

## Analysis by circle criterion

Backlash map from $\dot{\theta}_{\text{in}}$ to $\dot{\theta}_{\text{out}}$ is in the sector $[0, 1]$.

$-1/k_1 = \infty$ and $-1/k_2 = -1$

Hence closed loop is stable if $\operatorname{Re} G(i\omega) > -1$ for all $\omega$.

(For our motor example this proves stability when $K < 1$)

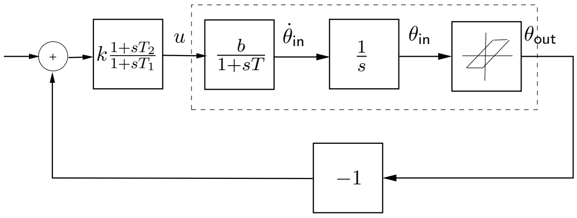
## Backlash compensation

- Mechanical solutions
- Dead-zone
- Linear controller design
- Backlash inverse

## Linear Controller Design

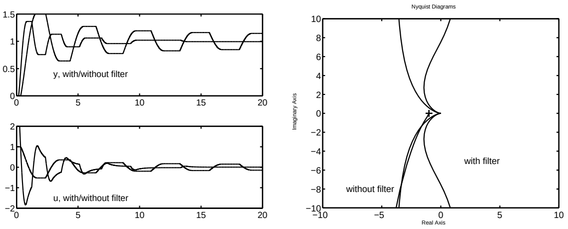
Introduce phase lead **to avoid** the $-1/N(A)$ curve:

Instead of only a P-controller we choose $K(s) = k\frac{1+sT_2}{1+sT_1}$



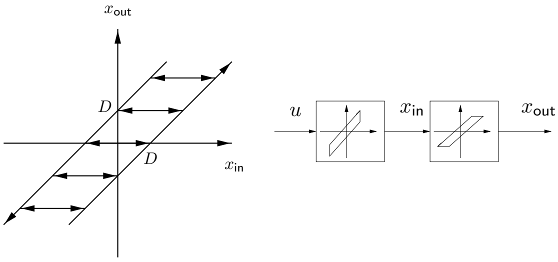
Controller $K(s) = k\frac{1+sT_2}{1+sT_1}$

Simulation with $T_1 = 0.5, T_2 = 2.0$



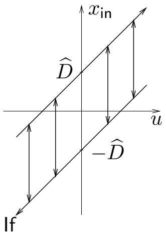
**No limit cycle, oscillation removed!**

## Backlash Inverse



Idea: Let $x_{\text{in}}$ jump $\pm 2D$ when $\dot{x}_{\text{out}}$ should change sign. Works if the backlash is directly on the system input!

## Backlash Inverse



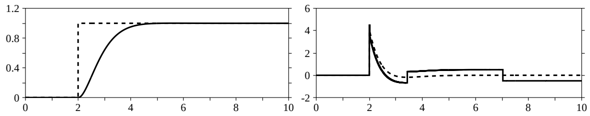
$$x_{\text{in}}(t) = \begin{cases} u + \widehat{D} & \text{if } u(t) > u(t-) \\ u - \widehat{D} & \text{if } u(t) < u(t-) \\ x_{\text{in}}(t-) & \text{otherwise} \end{cases}$$

If

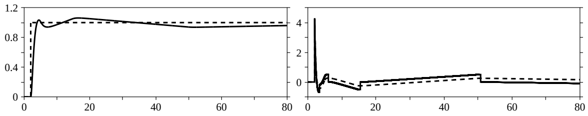
- $\widehat{D} = D$ then $x_{\text{out}}(t) = u(t)$ (perfect compensation)
- $\widehat{D} < D$: Under-compensation (decreased backlash)
- $\widehat{D} > D$: Over-compensation, often gives oscillations

## Example–Perfect compensation

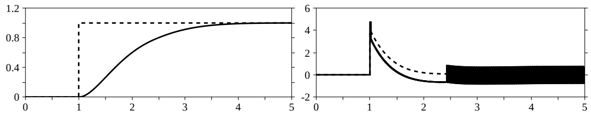
Motor with backlash on input, PD-controller



## Example–Under compensation



## Example–Over compensation



## Backlash–More advanced models

Warning: More detailed models needed sometimes

Model what happens when contact is attained

Model external forces that influence the backlash

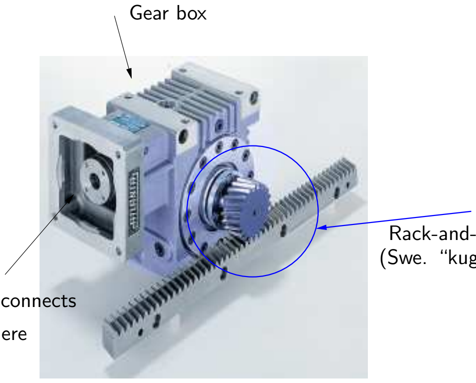
Model mass/moment of inertia of the backlash.

## Example: Parallel Kinematic Robot

Gantry-Tau robot:
Need backlash-free gearboxes for very high precision

EU-project: SMErobot™ `http://www.smerobot.org`

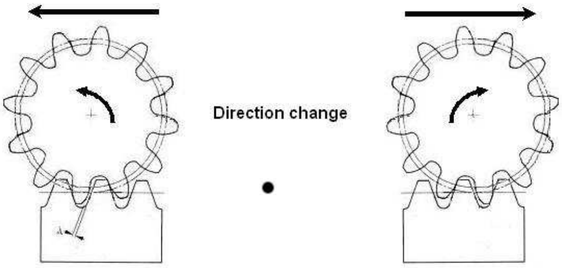
## "Rotational to Linear motion"


Gear box

Rack-and-pinion
(Swe. "kuggstng")

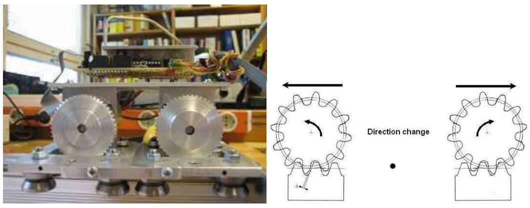
Motor connects
here

## Backlash in gearbox and rails



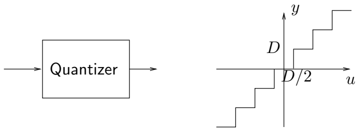
Remedy:
Use two motors in opposite directions: One motor can act as spring and brake to "reduce" backlash. Need measurements on both motor and arm-side.

## Backlash compensation



From master thesis by B. Brochier, *Control of a Gantry-Tau Structure, LTH, 2006*
See also master theses by j. Schiffer and L. Halt, 2009.

## Quantization



How accurate should the converters be? (8-14 bits?)

What precision is needed in computations? (8-64 bits?)

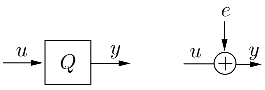
- Quantization in A/D and D/A converters
- Quantization of parameters
- Roundoff, overflow, underflow in operations

NOTE: Compare with **(different)** limits for "quantizer/dead-zone relay" in Lecture 6.

## Linear Model of Quantization

Model the quantization error as a stochastic signal $e$ independent of $u$ with rectangular distribution over the quantization size.

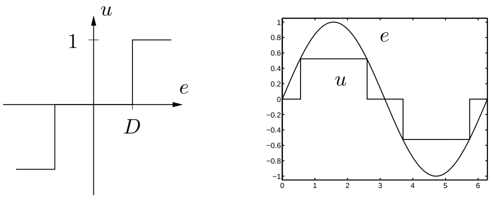
Works if quantization level is small compared to the variations in $u$



Rectangular noise distribution over $[-\frac{D}{2}, \frac{D}{2}]$ has the variance

$$Var(e) = \int_{-\infty}^{+\infty} e^2 f_e \, de = \int_{-D/2}^{D/2} e^2 \frac{1}{D} de = \frac{D^2}{12}$$

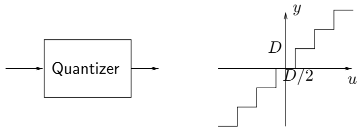
## Describing Function for Deadzone Relay



Lecture 6 ⇒

$N(A) = \frac{4}{\pi A}\sqrt{1 - D^2/A^2}$ for $A > D$ and zero otherwise

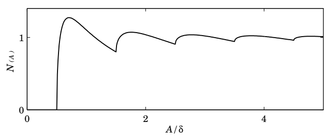
## Describing Function for Quantizer



$$N(A) = \begin{cases} 0 & A < \frac{D}{2} \\ \frac{4D}{\pi A}\sum_{k=1}^{n}\sqrt{1-\left(\frac{2k-1}{2A}D\right)^2} & \frac{2n-1}{2}D < A < \frac{2n+1}{2}D \end{cases}$$

(See exercise)

## Describing Function for Quantizer



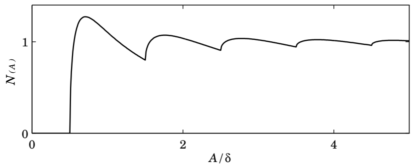
The maximum value is $4/\pi \approx 1.27$ for $A \approx 0.71D$.

Predicts limit cycle if Nyquist curve intersects negative real axis to the left of $-\pi/4 \approx -0.79$.

Should design for gain margin $> 1/0.79= 1.27$!

Note that reducing $D$ only reduces the size of the limit oscillation, the oscillation does not vanish completely.

## 5 minute exercise



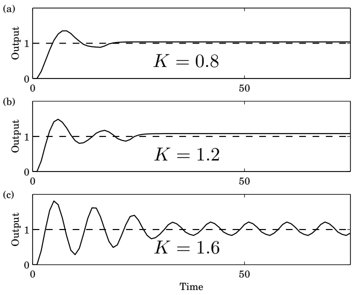
How does the shape of the describing function relate to the number of possible limit cycles and their stability.

What if the Nyquist plot

- intersects the negative real axis at $-0.80$?
- intersects the negative real axis at $-1$?
- intersects the negative real axis at $-2$?

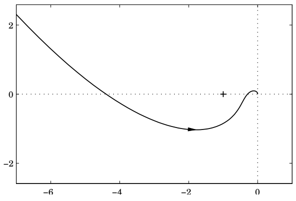
## Example – Motor with P-controller.

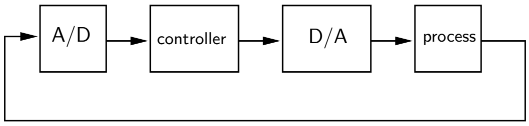
Roundoff at input, $D = 0.2$. Nyquist curve intersects at $-0.5K$. Hence stable for $K < 2$ without quantization. Stable oscillation predicted for $K > 2/1.27 = 1.57$.



## Example – Double integrator with 2nd order controller

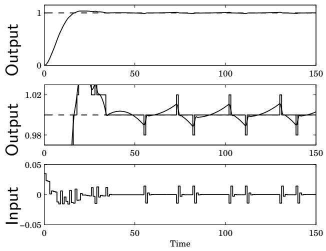
Nyquist curve





## Quantization at A/D converter

Double integrator with 2nd order controller, $D = 0.02$

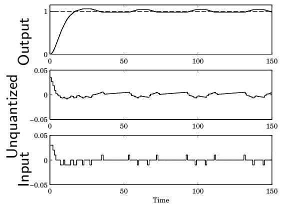


Describing function: $A_y \approx D/2 = 0.01$, period $T = 39$

Simulation: $A_y = 0.01$ and $T = 28$

## Quantization at D/A converter

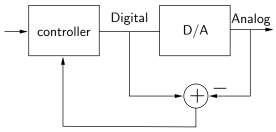
Double integrator with 2nd order controller, $D = 0.01$



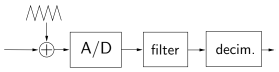
Describing function: $A_u \approx D/2 = 0.005$, period $T = 39$
Simulation: $A_u = 0.005$ and $T = 39$
Better prediction, since more sinusoidal signals

## Quantization Compensation

- Use improved converters, (smaller quantization errors/larger word length)
- Linear design, avoid unstable controller, ensure 1.3 gain margin
- Use the tracking idea from anti-windup to improve D/A converter

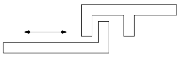


- Use analog dither, oversampling and digital low-pass filter to improve accuracy of A/D converter

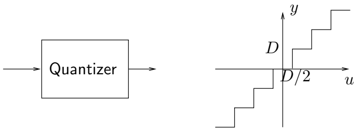


## Today's Goal

- To know models and compensation methods for backlash



- Be able to analyze the effect of quantization errors



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No More Lecture This Week!