**Lecture 4 — Lyapunov Stability**

### Material
- Glad & Ljung Ch. 12.2
- Khalil Ch. 4.1-4.3
- Lecture notes

### Main idea
Lyapunov formalized the idea:
*If the total energy is dissipated, then the system must be stable.*

**Main benefit:** By looking at how an energy-like function \( V \) (a so-called Lyapunov function) changes over time, we might conclude that a system is stable or asymptotically stable without solving the nonlinear differential equation.

**Analysis:** Check if \( V \) is decreasing with time:
- Continuous time: \( \frac{dV}{dt} < 0 \)
- Discrete time: \( V(k+1) - V(k) < 0 \)

**Main question:** How to find a Lyapunov function?

### Examples
Start with a Lyapunov candidate \( V \) to measure e.g.,
- "size"\(^1\) of state and/or output error,
- "size" of deviation from true parameters,
- energy difference from desired equilibrium,
- weighted combination of above
- ...

\(^1\)Often a magnitude measure or (squared) norm like \(|e|^2\), ...

### Stability Definitions
An equilibrium point \( x^* \) of \( \dot{x} = f(x) \) (i.e., \( f(x^*) = 0 \)) is
- **locally stable**, if for every \( R > 0 \) there exists \( r > 0 \), such that
  \[ \|x(0) - x^*\| < r \implies \|x(t) - x^*\| < R, \quad t \geq 0 \]
- **locally asymptotically stable**, if locally stable and
  \[ \|x(0) - x^*\| < r \implies \lim_{t \to \infty} x(t) = x^* \]
- **globally asymptotically stable**, if asymptotically stable for all \( x(0) \in \mathbb{R}^n \).

### Lyapunov Theorem for Local Stability
**Theorem** Let \( \dot{x} = f(x) \), \( f(x^*) = 0 \) where \( x^* \) is in the interior of \( \Omega \subset \mathbb{R}^n \). Assume that \( V : \Omega \to \mathbb{R} \) is a \( C^1 \) function. If

1. \( V(x^*) = 0 \)
2. \( V(x) > 0 \), for all \( x \in \Omega, x \neq x^* \)
3. \( \dot{V}(x) \leq 0 \) along all trajectories of the system in \( \Omega \)

then \( x^* \) is **locally stable**.

If also
4. \( \dot{V}(x) < 0 \) for all \( x \in \Omega, x \neq x^* \)

then \( x^* \) is **locally asymptotically stable**.

Furthermore, if \( \Omega = \mathbb{R}^n \) and also
5. \( V(x) \to \infty \) as \( \|x\| \to \infty \)

then \( x^* \) is **globally asymptotically stable**.
Lyapunov Functions (≈ Energy Functions)

A function \( V \) that fulfills (1)–(3) is called a Lyapunov function. Condition (3) means that \( V \) is non-increasing along all trajectories in \( \Omega \):
\[
V(x) = \frac{\partial V}{\partial x} \dot{x} = \sum_{i} \frac{\partial V}{\partial x_i} f_i(x) \leq 0
\]
where \( \frac{\partial V}{\partial x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \)

Geometric interpretation

Vector field points into sublevel sets
Trajectories can only go to lower values of \( V(x) \)

Lyapunov’s Linearization Method

Consider \( \dot{x} = f(x) \)
Assume that \( f(0) = 0 \).
\[
\dot{x} = Ax + g(x) , \quad ||g(x)|| = o(||x||) \text{ as } x \to 0.
\]
Find Lyapunov function for linearization \( \dot{x} \). Use it for original system (near the equilibrium)!

Conservation and Dissipation

Conservation of energy: \( \dot{V}(x) = \frac{\partial V}{\partial x} f(x) = 0 \), i.e., the vector field \( f(x) \) is everywhere orthogonal to the normal \( \frac{\partial V}{\partial x} \) to the level surface \( V(x) = c \).

Example: Total energy of a lossless mechanical system or total fluid in a closed system.

Dissipation of energy: \( \dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \), i.e., the vector field \( f(x) \) and the normal \( \frac{\partial V}{\partial x} \) to the level surface \( \{ z : V(z) = c \} \) make an obtuse angle (Sw. “trubbig vinkel”).

Example: Total energy of a mechanical system with damping or total fluid in a system that leaks.

More matrix results

- for symmetric matrix \( M = M^T \)
  \[
  \lambda_{\min}(M)||x||^2 \leq x^T M x \leq \lambda_{\max}(M)||x||^2 , \quad \forall x
  \]
  Proof idea: factorize \( M = U \Lambda U^T \), unitary \( U \) (i.e., \( ||Ux|| = ||x|| \forall x \)), \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \)

- for any matrix \( M \)
  \[
  \|Mx\| \leq \sqrt{\lambda_{\max}(M^T M)} ||x|| , \quad \forall x
  \]

Lyapunov Stability for Linear Systems

Linear system: \( \dot{x} = Ax \)

Lyapunov equation: Let \( Q = Q^T > 0 \). Solve
\[
PA + A^T P = -Q
\]
with respect to the symmetric matrix \( P \).

Lyapunov function: \( V(x) = x^T P x , \Rightarrow 
\)
\[
V(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x < 0
\]

Asymptotic Stability: If \( P = P^T > 0 \), then the Lyapunov Stability Theorem implies (local–global) asymptotic stability, hence the eigenvalues of \( A \) must satisfy \( \text{Re} \lambda_k(A) < 0 \), \forall k
Converse Theorem for Linear Systems

If Re λk(A) < 0 ∀k, then for every Q = Q^T > 0 there exists P = P^T > 0 such that PAP = −Q

Proof: Choose P = \int_0^\infty e^{AT}Qe^{A^T}dt. Then
\[ PAP = \lim_{t \to \infty} \int_0^t (A^T e^{AT}Qe^{A^T} + e^{AT}Qe^{A^T}) dt \]
\[ = \lim_{t \to \infty} [e^{AT}Qe^{A^T}]^t_0 \]
\[ = −Q \]

Example: Lyapunov function for linear system

\[ \dot{x} = Ax = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} x \] (1)

Eigenvalues of A : {−1, −3} ⇒ (global) asymptotic stability.

Find a quadratic Lyapunov function for the system (1):
\[ V(x) = x^TPx = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad P = P^T > 0 \]

Take any Q = Q^T > 0 , say Q = I_{2×2}. Solve A^TP + PA = −Q.

Phase plot showing that
\[ V = \frac{1}{2}(x_1^2 + x_2^2) = \begin{bmatrix} x_1 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \] does NOT work.

Lyapunov’s Linearization Method

Recall from Lecture 2:

Theorem Consider
\[ \dot{x} = f(x) \]
Assume that f(0) = 0. Linearization
\[ \dot{x} = Ax + g(x), \quad ||g(x)|| = o(||x||) \] as \( x \to 0 \).

(1) Re λk(A) < 0, ∀k ⇒ \( x = 0 \) locally asympt. stable
(2) \( \exists k : \text{Re}\lambda_k(A) > 0 \) ⇒ \( x = 0 \) unstable

Proof of (1) in Lyapunov’s Linearization Method

Put \( V(x) := x^TPx \). Then, \( V(0) = 0, V(x) > 0 \forall x \neq 0 \), and
\[ \dot{V}(x) = x^TPf(x) + f^T(x)Px \]
\[ = x^TP(Ax + g(x)) + ||x||^2 A^TP + P^Tg(x)Px \]
\[ = x^T(PA + A^TP)x + 2x^TPg(x) = −x^TQx + 2x^TPg(x) \]
\[ x^TQx ≥ \lambda_{\text{min}}(Q)||x||^2 \]

and for all \( γ > 0 \) there exists \( r > 0 \) such that
\[ ||g(x)|| < γ||x||, \quad \forall||x|| < r \]

Thus, choosing \( γ \) sufficiently small gives
\[ \dot{V}(x) ≤ −(\lambda_{\text{min}}(Q) − 2γ\lambda_{\text{max}}(P)||x||^2 < 0 \]
Lyapunov Theorem for Global Asymptotic Stability

**Theorem** Let $\dot{x} = f(x)$ and $f(x^*) = 0$. If there exists a $C^1$ function $V: \mathbb{R}^n \to \mathbb{R}$ such that

1. $V(x^*) = 0$
2. $V(x) > 0$, for all $x \neq x^*$
3. $\dot{V}(x) < 0$ for all $x \neq x^*$
4. $V(x) \to \infty$ as $\|x\| \to \infty$

then $x^*$ is a globally asymptotically stable equilibrium.

Radial Unboundedness is Necessary

If the condition $V(x) \to \infty$ as $\|x\| \to \infty$ is not fulfilled, then global stability cannot be guaranteed.

**Example** Assume $V(x) = x^2/(1 + x^2) + x^2$ is a Lyapunov function for a system. Can have $\|x\| \to \infty$ even if $\dot{V}(x) < 0$.

Somewhat Stronger Assumptions

**Theorem:** Let $\dot{x} = f(x)$ and $f(x^*) = 0$. If there exists a $C^1$ function $V: \mathbb{R}^n \to \mathbb{R}$ and numbers $\alpha, \tau, c > 0$ such that

1. $V(x^*) = 0$
2. $V(x) > \varepsilon \|x - x^*\| > 0$ for all $x \neq x^*$
3. $\dot{V}(x) \leq -\alpha V(x)$ for all $x$
4. $V(x) \to \infty$ as $\|x\| \to \infty$

then $x^*$ is globally exponentially stable.

Proof Idea

Assume $x(t) \neq 0$ (otherwise we have $x(t) = 0$ for all $t > 0$). Then

$$\frac{\dot{V}(x)}{V(x)} \leq -\alpha$$

Integrating from 0 to $t$ gives

$$\log V(x(t)) - \log V(x(0)) \leq -\alpha t \Rightarrow V(x(t)) \leq e^{-\alpha t} V(x(0))$$

Hence, $V(x(t)) \to 0$ exponentially as $t \to \infty$.

From (2) it follows that also $|x(t) - x^*|$ decays exponentially.

Invariant Sets

**Definition:** A set $M$ is called invariant if for the system $\dot{x} = f(x)$,

$x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.

LaSalle’s Invariant Set Theorem

**Theorem** Let $\Omega \subseteq \mathbb{R}^n$ compact invariant set for $\dot{x} = f(x)$. Let $V: \Omega \to \mathbb{R}$ be a $C^1$ function such that $\dot{V}(x) \leq 0, \forall x \in \Omega, M := \{x \in \Omega : \dot{V}(x) = 0\}$, $E := \{x \in \Omega : \dot{V}(x) = 0\}$, $M := \text{largest invariant subset of} E$ $\implies \forall x(0) \in \Omega, x(t) \text{ approaches } M \text{ as } t \to +\infty$

Note that $V$ must not be a positive definite function in this case.

A Motivating Example (revisited)

$$\begin{align*}
\dot{x} & = -b\dot{x} - k_0 x - k_1 x^3 \\
V(x, \dot{x}) & = (2m\dot{x}^2 + 2k_0 x^2 + k_1 x^4)/4 > 0, \quad V(0, 0) = 0 \\
\dot{V}(x, \dot{x}) & = -b\dot{x}^2 \text{ gives } E = \{x, \dot{x} : \dot{x} = 0\}
\end{align*}$$

Assume there exists $(\tilde{x}, \tilde{\dot{x}}) \in M$ such that $\tilde{x}(t_0) \neq 0$. Then

$$m\ddot{x}(t_0) = -b\dot{x}(t_0) - k_1 x^2(t_0) \neq 0$$

so $\dot{x}(t_0) \neq 0$ so the trajectory will immediately leave $M$. A contradiction to that $M$ is invariant.

Hence, $M = \{(0, 0)\}$ so LaSalle’s theorem gives global asymptotic stability of the origin.

Special Case: Global Stability of Equilibrium

**Theorem:** Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $C^1$ function $V: \mathbb{R}^n \to \mathbb{R}$ such that

1. $V(0) = 0, V(x) > 0$ for all $x \neq 0$
2. $\dot{V}(x) \leq 0$ for all $x$
3. $V(x) \to \infty$ as $\|x\| \to \infty$
4. The only solution of $\dot{x} = f(x)$, $\dot{V}(x) = 0$ is $x(t) = 0 \forall t$

$\implies x = 0$ is globally asymptotically stable.
Stability analysis using input-output (frequency) method

Introduce $\tilde{\Omega} = \{x : \|x\| = 1\}$

Let $V = \frac{1}{2}(\tilde{x}^2 + \gamma a \tilde{a}^2 + \gamma b \tilde{b}^2)$

$\dot{V} = \tilde{x} \tilde{x} + \gamma a \tilde{a} \dot{\tilde{a}} + \gamma b \tilde{b} \dot{\tilde{b}} = -2(\tilde{x}^2 + \tilde{a}^2 + \tilde{b}^2)$

where the last equality follows if we choose

$\dot{\tilde{a}} = -\tilde{a} = \frac{1}{\gamma a} \tilde{x}$

$\dot{\tilde{b}} = -\tilde{b} = -\frac{1}{\gamma b} \tilde{x}$

Invariant set: $\tilde{x} = 0$.

This proves that $\tilde{x} \to 0$.

(The parameters $\tilde{a}$ and $\tilde{b}$ do not necessarily converge: $u \equiv 0$.)

Demonstration if time permits

Estimation of parameters starts at $t=10$ s.

Next Lecture

- Stability analysis using input-output (frequency) methods