Lecture 3

- Phase-plane analysis
- Classification of singularities
- Stability of periodic solutions

Material

- Glad and Ljung: Chapter 13
- Khalil: Chapter 2.1–2.3
- Lecture notes

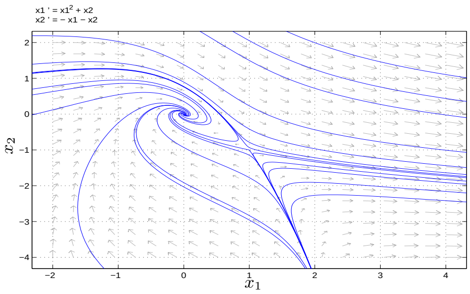
Today's Goal

You should be able to

- *sketch phase portraits for two-dimensional systems*
- *classify equilibria into nodes, focus, saddle points, and center points.*
- *analyze limit cycles through Poincaré maps*

First glipse of phase plane portraits: Consider the system

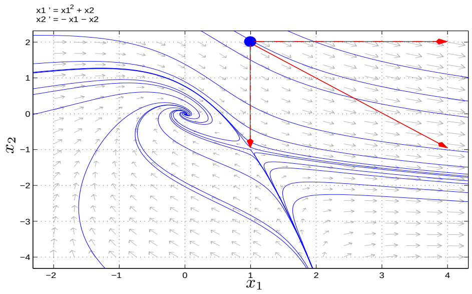
$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = -x_1 - x_2$$



Flow-interpretation: To each point (x_1, x_2) in the plane there is an associated flow-direction $\frac{dx}{dt} = f(x_1, x_2)$

First glipse of phase plane portraits: Consider the system

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = -x_1 - x_2$$



In the point $(x_1, x_2) = (1, 2)$ **the vector field** is pointing in the direction $(1^2 + 2, -1 - 2) = (3, -3)$.

Linear Systems Revival

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analytic solution: $x(t) = e^{At}x(0)$.

If A is diagonalizable, then

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where v_1, v_2 are the eigenvectors of A $\quad (Av_1 = \lambda_1 v_1$ etc).

Matlab:

```
>> [V,Lambda]=eig(A)
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Example: Two real negative eigenvalues

Given the eigenvalues $\underbrace{\lambda_1}_{\text{faster}} < \underbrace{\lambda_2}_{\text{slower}} < 0$, with corresponding eigenvectors v_1 and v_2, respectively.

Solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Fast eigenvalue/vector: $x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$ for small t.
Moves along the fast eigenvector for small t

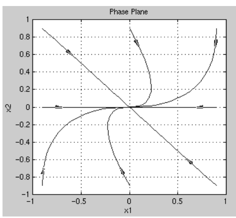
Slow eigenvalue/vector: $x(t) \approx c_2 e^{\lambda_2 t} v_2$ for large t.
Moves along the slow eigenvector towards $x = 0$ for large t

Example—Stable Node

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$$(\lambda_1, \lambda_2) = (-1, -2) \quad \text{and} \quad \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

v_1 is the slow direction and v_2 is the fast.



Example—Unstable Focus

$$\dot{x} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x, \qquad \sigma, \omega > 0, \qquad \lambda_{1,2} = \sigma \pm i\omega$$

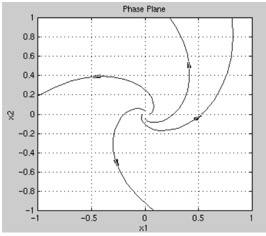
$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\sigma t}e^{i\omega t} & 0 \\ 0 & e^{\sigma t}e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} x(0)$$

In polar coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan x_2/x_1$ ($x_1 = r\cos\theta$, $x_2 = r\sin\theta$):

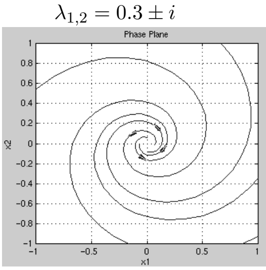
$$\dot{r} = \sigma r$$
$$\dot{\theta} = \omega$$

Example- unstable focus cont'd

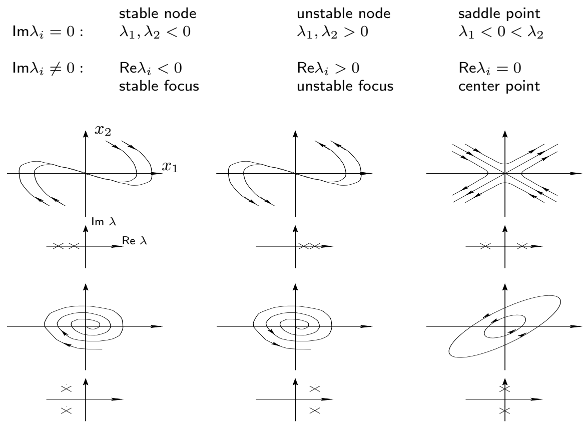
$\lambda_{1,2} = 1 \pm i$



$\lambda_{1,2} = 0.3 \pm i$



Equilibrium Points for Linear Systems



4 minute exercise

What is the phase portrait if $\lambda_1 = \lambda_2$?

Linear Time-Varying Systems (warning)

Warning: Pointwise "Left Half-Plane eigenvalues" of $A(t)$ (*i.e., time-varying systems*) do *NOT* impose stability!!!

$$A(t) = \begin{pmatrix} -1 + \alpha \cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Pointwise eigenvalues are given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

which are in the LHP for $0 < \alpha < 2$ (and here even constant). However,

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t} \cos t & e^{-t} \sin t \\ -e^{(\alpha-1)t} \sin t & e^{-t} \cos t \end{pmatrix} x(0),$$

which is an unbounded solution for $\alpha > 1$.

Phase-Plane Analysis for Nonlinear Systems

Close to equilibria "nonlinear system" $\approx$ "linear system".

Theorem Assume

$$\dot{x} = f(x)$$

is linearized at x_0 so that

$$\dot{x} = Ax + g(x),$$

where $g \in C^1$ and $\frac{g(x) - g(x_0)}{\|x - x_0\|} \to 0$ as $x \to x_0$.

If $\dot{z} = Az$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

If the linearized system has a center, then the nonlinear system has either a center or a focus.

How to Draw Phase Portraits

If done by hand then

1. Find equilibria (also called singularities)
2. Sketch local behavior around equilibria
3. Sketch $(\dot{x}_1, \dot{x}_2)$ for some other points. Use that $\frac{dx_1}{dx_2} = \frac{\dot{x}_1}{\dot{x}_2}$.
4. Try to find possible limit cycles
5. Guess solutions

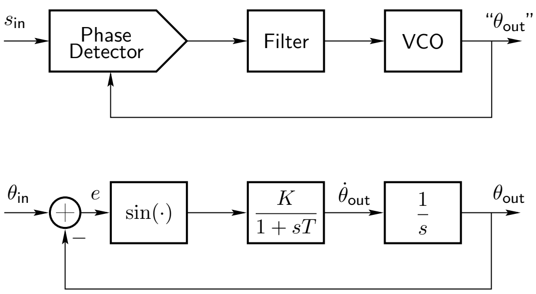
Matlab: `pptool6/pptool7, dfield6/dfield7, dee, ICTools`, etc.

PPTool and some other tools for Matlab is available on or via

`http://www.control.lth.se/course/FRTN05`

Phase-Locked Loop

A PLL tracks phase $\theta_{\text{in}}(t)$ of a signal $s_{\text{in}}(t) = A \sin[\omega t + \theta_{\text{in}}(t)]$.



Singularity Analysis of PLL

Let $x_1(t) = \theta_{\text{out}}(t)$ and $x_2(t) = \dot{\theta}_{\text{out}}(t)$.
Assume $K, T > 0$ and $\theta_{\text{in}}(t) = \theta_{\text{in}}$ constant.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -T^{-1} x_2 + K T^{-1} \sin(\theta_{\text{in}} - x_1)$$

Singularities are $(\theta_{\text{in}} + n\pi, 0)$, since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$
$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{in} - x_1) = 0 \Rightarrow x_1 = \theta_{\text{in}} + n\pi$$

Singularity Classification of Linearized System

Linearization gives the following characteristic equations:

n even:

$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

$K > (4T)^{-1}$ gives stable focus
$0 < K < (4T)^{-1}$ gives stable node

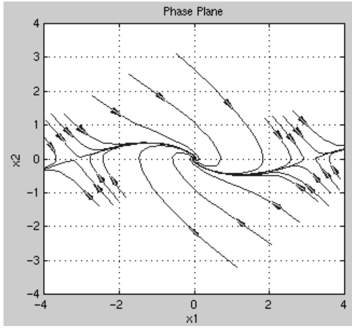
n odd:

$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all $K, T > 0$

Phase-Plane for PLL

$K = 1/2$, $T = 1$: Focus $(2k\pi, 0)$, saddle points $((2k+1)\pi, 0)$



Summary

Phase-plane analysis limited to second-order systems (sometimes it is possible for higher-order systems to fix some states)

Many dynamical systems of order three and higher not fully understood (chaotic behaviors etc.)

Bonus — Discrete Time

Many results are parallel (observability, controllability,...)

Example: The difference equation

$$x_{k+1} = f(x_k)$$

is asymptotically stable at x^* if the linearization

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} \text{ has all eigenvalues in } |\lambda| < 1$$

(that is, within the unit circle).

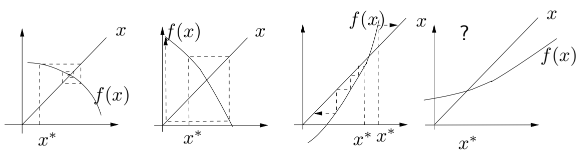
Example (cont'd): Numerical iteration

$$x_{k+1} = f(x_k)$$

to find fixed point

$$x^* = f(x^*)$$

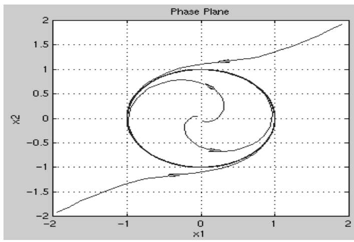
When does the iteration converge?



Periodic Solutions: $x(t+T) = x(t)$

Example of an asymptotically stable periodic solution:

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \qquad (1)$$



Periodic solution: Polar coordinates.

Let

$$x_1 = r\cos\theta \quad \Rightarrow dx_1 = \cos\theta dr - r\sin\theta d\theta$$
$$x_2 = r\sin\theta \quad \Rightarrow dx_2 = \sin\theta dr + r\cos\theta d\theta$$

$\Rightarrow$

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now

$$\dot{x}_1 = r(1-r^2)\cos\theta - r\sin\theta$$
$$\dot{x}_2 = r(1-r^2)\sin\theta + r\cos\theta$$

which gives

$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$

Only $r = 1$ is a stable equilibrium!

A system has a **periodic solution** if for some $T > 0$

$$x(t+T) = x(t), \quad \forall t \geq 0$$

Note that a constant value for $x(t)$ by convention not is regarded as periodic.

- When does a periodic solution exist?
- When is it locally (asymptotically) stable? When is it globally asymptotically stable?

Poincaré map ("Stroboscopic map")

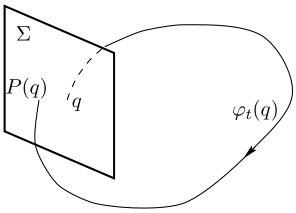
$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n$$

$\varphi_t(q)$ is the solution starting in q after time t.

$\Sigma \subset \mathbf{R}^{n-1}$ is a hyperplane transverse to φ_t.

The Poincaré map $P : \Sigma \to \Sigma$ is

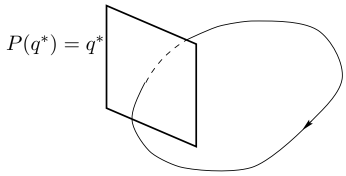
$$P(q) = \varphi_{\tau(q)}(q), \qquad \tau(q) \text{ is the first return time}$$



Limit Cycles

If a simple periodic orbit pass through q^*, then $P(q^*) = q^*$.

Such an orbit is called a *limit cycle*.
q^* is called a *fixed point* of P.



Does the iteration $q_{k+1} = P(q_k)$ converge to q^*?

Locally Stable Limit Cycles

The linearization of P around q^* gives a matrix $W = \frac{\partial P}{\partial q}\Big|_{q^*}$ so

$$(q_{k+1} - q^*) \approx W(q_k - q^*),$$

if q_k is close to q^*.

- If all $|\lambda_i(W)| < 1$, then the corresponding limit cycle is locally **asymptotically stable**.
- If $|\lambda_i(W)| > 1$, then the limit cycle is **unstable**.

Linearization Around a Periodic Solution

The linearization of

$$\dot{x}(t) = f(x(t))$$

around $x_0(t) = x_0(t+T)$ is

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$$
$$A(t) = \frac{\partial f}{\partial x}(x_0(t)) = A(t+T)$$

P is the map from the solution at $t = 0$ to $t = \tau(q)$.

Example—Stable Unit Circle

Rewrite (1) in polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Choose $\Sigma = \{(r, \theta) : r > 0, \theta = 2\pi k\}$.

The solution is

$$\varphi_t(r_0, \theta_0) = \left([1 + (r_0^{-2} - 1)e^{-2t}]^{-1/2}, t + \theta_0\right)$$

First return time from any point $(r_0, \theta_0) \in \Sigma$ is $\tau(r_0, \theta_0) = 2\pi$.

Example—Stable Unit Circle

The Poincaré map is

$$P(r_0) = [1 + (r_0^{-2} - 1)e^{-2 \cdot 2\pi}]^{-1/2}$$

$r_0 = 1$ is a fixed point.

The limit cycle that corresponds to $r(t) = 1$ and $\theta(t) = t$ is locally asymptotically stable, because

$$W = \frac{dP}{dr_0}(1) = \left[e^{-4\pi}\right]$$

and

$$|W| = \left|\frac{dP}{dr_0}(1)\right| = |e^{-4\pi}| < 1$$

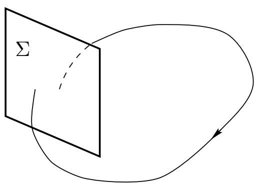
Example—The Hand Saw

Can we stabilize the inverted pendulum by vertical oscillations?

The Hand Saw—Poincaré Map

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{\ell}\left(g + a\omega^2 \sin x_3\right) \sin x_1$$
$$\dot{x}_3(t) = \omega$$

Choose $\Sigma = \{x_3 = 2\pi k\}$.



The Hand Saw–Poincaré Map

$q^* = 0$ and $T = 2\pi/\omega$. No explicit expression for P. It is, however, easy to determine W numerically. Do two (*or preferably many more*) different simulations with different, small, initial conditions $x(0) = y$ and $x(0) = z$.
Solve W through (*least squares solution of*)

$$\begin{pmatrix} x(T)\Big|_{x(0)=y} & x(T)\Big|_{x(0)=z} \end{pmatrix} = W \begin{pmatrix} y & z \end{pmatrix}$$

This gives for $a = 1$cm, $\ell = 17$cm, $\omega = 180$

$$W = \begin{pmatrix} 1.37 & 0.035 \\ -3.86 & 0.630 \end{pmatrix}$$

which has eigenvalues $(1.047, 0.955)$. Unstable.

W is stable for $\omega > 183$

The Hand Saw—Stability Condition

Make the assumptions that

$$\ell \gg a \qquad \text{and} \qquad a\omega^2 \gg g$$

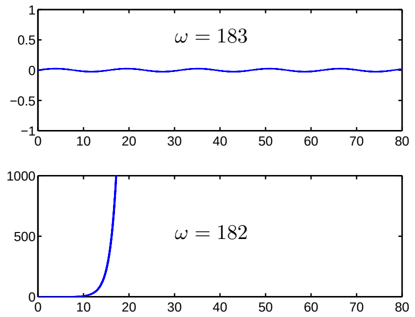
Then some calculations show that the Poincaré map is stable at $q^* = 0$ when

$$\omega > \frac{\sqrt{2g\ell}}{a}$$

$a = 1$ cm and $\ell = 17$ cm give $\omega > 182.6$ rad/s (29 Hz).

The Hand Saw—Simulation

Simulation results give good agreement



Next Lecture

- Lyapunov methods for stability analysis

Lyapunov generalized the idea of: *If the total energy is dissipated along the trajectories (i.e the solution curves), the system must be stable.*

Benefit: Might conclude that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.