Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - May 28, 2008, 2–7 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each sub-problem. Most sub-problems can be solved independently of each other.
Preliminary grades:

3: $12 - 16$ points

4: $16.5 - 20.5$ points

5: $21 - 25$ points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik". Pocket calculator.

Results

The exam results will be posted within two weeks after the day of the exam on the notice-board at the Department. Contact the lecturer Anders Robertsson for checking your corrected exam.

Note!

In many cases the sub-problems can be solved independently of each other.

Good Luck!

Solutions to the exam in **Nonlinear Control and Servo Systems** (FRTN05)
May, 2008.

1. Consider the control system

$$\ddot{x} - 2(\dot{x})^2 + x = u - 1 \tag{1}$$

a. Write the system in first-order state-space form. (1 p)

b. Suppose $u(t) \equiv 0$. Find and classify (using linearization) all equilibria and determine if they are stable or asymptotically stable if possible. Discuss if the stability results are global or local. (2 p)

c. Show that Eq. (1) satisfies the periodic solution $x(t) = cos(t)$, $u(t) = cos(2t)$. (1 p)

d. Design a state-feedback controller $u = u(x, \dot{x})$ for (1), such that the origin of the closed loop system is globally asymptotically stable. (1 p)

Solution

a. Introduce $x_1 = x$, $x_2 = \dot{x}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 2x_2^2 + u - 1 \end{aligned} \tag{2}$$

b. Let $\dot{x}_1 = \dot{x}_2 = 0 \Rightarrow (x_1, x_2) = (-1, 0)$ is the only equilibrium. The linearization around this point is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 4x_2 \end{bmatrix}_{(x_1^o, x_2^o) = (-1, 0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The characteristic equation for the linearized system is $s^2 + 1 = 0 \Rightarrow s = \pm i$. In general linearization only gives local behaviour of the nonlinear system, but as the linaerized sytem has a center point we can not conclude even local stability of the nonlinear system from this.

c.

$$x = \cos(t) \Rightarrow \dot{x} = -\sin(t) \Rightarrow \ddot{x} = -\cos(t)$$

By inserting this in the system dynamics and using *e.g.*, $u = \cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$ we get

$$\ddot{x} - 2(\dot{x})^2 + x = -\cos(t) - 2\sin^2(t) + \cos(t) = 2 + cos^2(t) - 2 = u - 1$$

which shows that the trajectory is a solution.
The linearized system is thus

$$\begin{aligned} \delta\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 4x_2 \end{bmatrix}_{(x_1^o, x_2^o) = (\cos(t), -\sin(t))} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \\ &= \begin{bmatrix} 0 & 1 \\ -1 & -4\sin(t) \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \end{aligned} \tag{3}$$

where

$$\delta x = \begin{bmatrix} x_1(t) - cos(t) \\ x_2(t) - (-\sin(t)) \end{bmatrix}, \quad \delta u = u(t) - \cos(2t)$$

d. The simplest way is to cancel the constant term and the nonlinearity with the control signal and introduce some linear feedback.

$$u = +1 - 2(\dot{x}_2)^2 - a\dot{x}, \quad a > 0 \Rightarrow \ddot{x} + a\dot{x} + x = 0$$

As the resulting system is linear and time invariant with poles in the left half plane for all $a > 0$ it is GAS.

2. Consider the system in Figure 1.

a. Introduce states and find all equilibrium points of the system. (1.5 p)

b. *S*ketch the vector field locally around one of the equilibrium points of the system in a phase plane plot. (1 p)

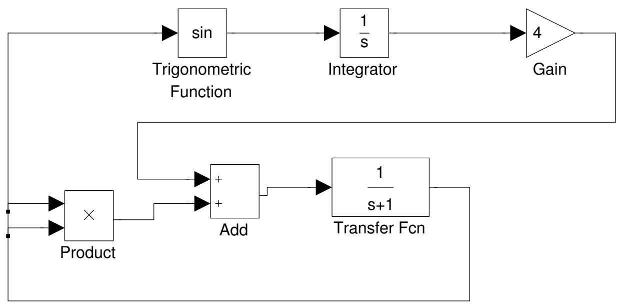

Figure 1 System in Problem 2.

Solution

a. Set x_1 after "Integrator" and x_2 after "Transfer Fcn". The signal entering the "Transfer Fcn"-block is $4x_1 + x_2^2$. The state equations become

$$\dot{x_1} = sin(x_2) \tag{4}$$
$$\dot{x_2} = -x_2 + (4x_1 + x_2^2) \tag{5}$$

The equilibrium points are determined by setting the derivatives to 0. The first equation gives $sin(x_2) = 0$ which gives that $x_2 = k\pi$ where $k = ... -2, -1, 0, 1, 2...$ at the equilibrium points. The second equation gives $x_1 = x_2(1 - x_2)/4 = k\pi(1 - k\pi)/4$ at the equilibrium points.

b. The origin is an equilibrium point (saddle point). A phase plane plot of the system is found in Figure 2.

3. We want to design an oscillator by the interconnection of a first order system with time delay, and a relay, see Figure 3. The system $G(s) = \dfrac{k}{s+1} \cdot e^{-Ls}$ and $\varphi(\cdot)$ is a relay with amplitude 1 (*i.e.,* $\varphi(z) = sign(z)$). We want to achieve an oscillation with amplitude = 2 Volts and a frequency of 5 Hz. Determine the parameters $k > 0$ and $L > 0$ to achieve this. Will the oscillation be stable? Motivate your answer. (3 p)

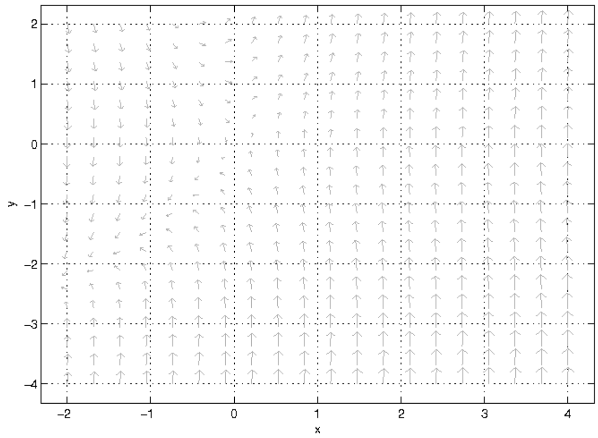

Figure 2 Phase plane plot for problem 2

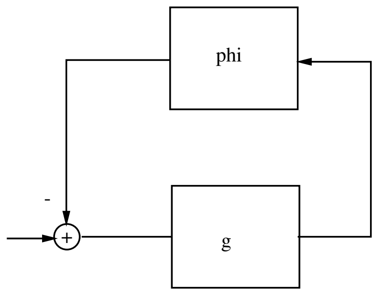

Figure 3 The block diagram for the oscillator system in Problem 3. The nonlinearity $\varphi(\cdot)$ is a relay with amplitude 1

Solution

The describing function for a relay which switches between -1 and 1 is $N(A) = \dfrac{4}{A\pi}$. Find k and L such that $G(j\omega) = -1/N(A)$. $A = 2$ and $\omega = 2\pi \cdot 5 = 10\pi$

$$arg\{G(j\omega)\} = -atan(\omega/1) - \omega L = -\pi \Rightarrow L \approx 0.05$$

$$|G(j\omega)| = k/\sqrt{\omega^2 + 1^2} = \pi/2 \Rightarrow k \approx 50$$

It will be a stable limit cycle (see course literature for condition).

4. An exponentially stable linear system $G(s)$ is negative feedback interconnected with a nonlinearity ψ. The Nyquist diagram of the linear system is shown in Figure 4. (Note: For your answer it is more important to clearly mark in a figure where you get your data from than to have all digits correct.)

a. What is the larges sector $\psi \in [-k, k]$ for which *the small gain theorem* guarantees stability for the closed loop? (1 p)

b. What is the largest sector $\psi \in [0, \beta]$ for which *the circle criterion* guarantees stability for the closed loop? (1 p)

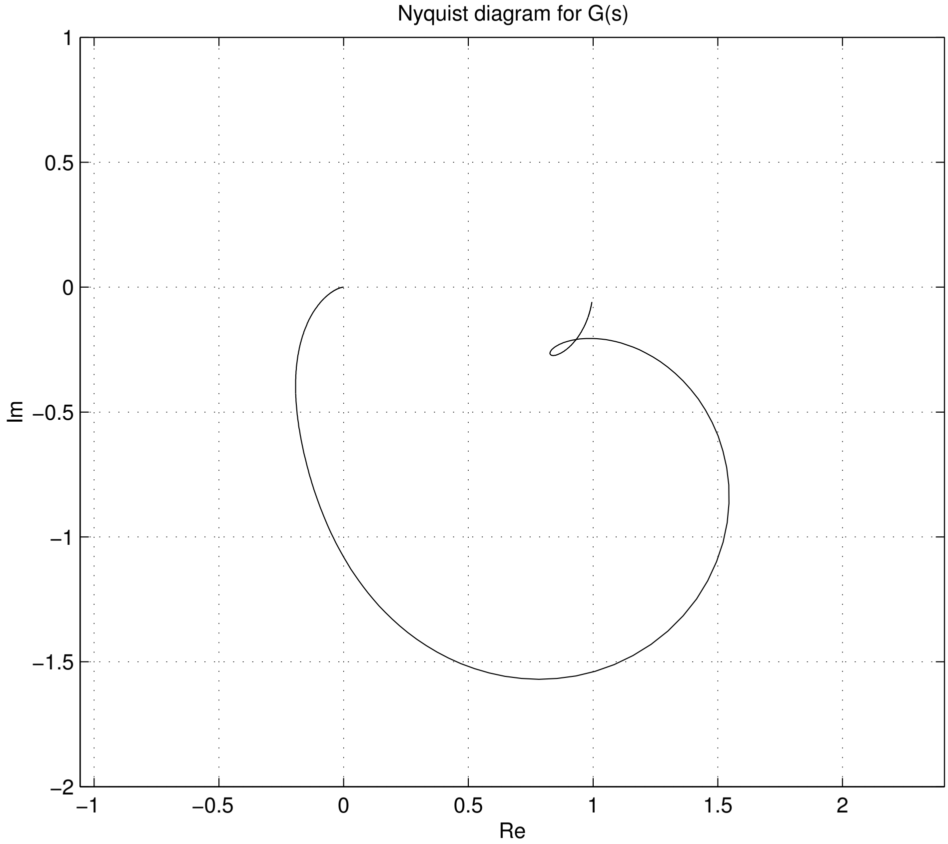

Figure 4 Nyquist diagram for linear system $G(s)$ in Problem 4.

c. What if you know that the nonlinearity is negative and upper bounded by 0? Find the largest sector $\psi \in [\alpha, 0]$, where $\alpha < 0$ for which *the circle criterion* guarantees stability for the closed loop. (1 p)

Solution

a. In this case we first want to find the maximum gain of the linear system which equals the largest magnitude ('radius') of the Nyquist curve. From the Nyquist curve we see that this is about 2. The small gain theorem then allows the sector to have $k < 1/2 = 0.5$.

b. According to the circle criterion, in this case the closed loop will be stable for the nonlinearity in the sector $[0, \beta]$ if the Nyquist curve stays to the right of the vertical line $-1/\beta$. From the Nyquist curve we see that we can take $\beta \approx 1/0.25 = 4$.

c. Multiply the nonlinearity and the system by -1 and apply the "ordinary" circle criterion. This means that the mirrored Nyquist curve must stay to the right of $-1/(-\alpha)$. The mirrored Nyquist curve is to the right of the vertical line -1.6, which means we can choose $\alpha = -1/1.6$

5. Use Lyapunov theory to prove that the system

$$\begin{aligned}\dot{x} &= -x - 2y^2 \\ \dot{y} &= xy - y^3\end{aligned}$$

is globally asymptotically stable. (2 p)

Solution

One choice is the Lyapunov function

$$V(x, y) = x^2 + 2y^2.$$

Then

$$\frac{d}{dt}V = 2x\dot{x} + 4y\dot{y} = -2x^2 - 4xy^2 + 4xy^2 - 4y^4 = -2x^2 - 4y^4 < 0, \quad (x, y) \neq 0.$$

As V is positive and radially unbounded this proves global asymptotic stability of the system.

Another choice of Lyapunov function is

$$V(x, y) = \frac{1}{2}(x^2 + y^2).$$

This gives

$$\frac{d}{dt}V = x\dot{x} + y\dot{y} = -(x^2 + xy^2 + y^4) = -(x + \frac{1}{2}y^2)^2 - \frac{3}{4}y^4 < 0, \quad (x, y) \neq 0$$

which also proves global asymptotic stability of the system.

6. The famous control engineer Wanda B. Stable has tried to stabilize the system

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 + x_2 u_1.\end{aligned}$$

for several days without success. In her desperation she has consulted her Swedish friend Anna Lys. *"—I am desperate. I have tried everything: linearization, Lyapunov theory, sliding . . . It just doesn't seem to be possible. What do you say, Anna?"*. What should Anna answer? Is it possible to find a control law $u = [u_1(x), u_2(x)]$ so that the origin $x = 0$ is made locally asymptotically stable? *(Hint: Consider the function $H = x_3 - x_1 x_2$ and see how it evolves with time.)* (2 p)

Solution

We have $\dot{H} = 0$ so $H(x(t)) = H(x(t_0))$, no matter how $x(t)$ evolves, and no matter how we chose u_1 and u_2. Note that $H(0) = 0$.

Now assume that the system is initiated near the origin with $H(x(t_0)) \neq 0$. Since $\dot{H} = 0$, the system will never reach $H(x(t)) = 0$, and therefore the states will never be asymptotically stabilized in $x = 0$.

7. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2^2 - 2\ \mathrm{sign}(x_1 + x_2) + u \\ \dot{x}_2 &= x_1\end{aligned}$$

a. Set u=0 and calculate the sliding surface. Also determine if the dynamics on the sliding surface is stable. (2 p)

b. Design a **continuous** controller, u, that brings all solution to the switching line $x_1 + x_2 = 0$. Does this control change the behaviour on the switching line? (1.5 p)

Solution

a. The dynamics are

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2^2 - 2\ \mathrm{sign}(x_1 + x_2) \\ \dot{x}_2 &= x_1\end{aligned}$$

set $\sigma(x) = x_1 + x_2$ and use equivalent control to calculate the sliding surface. Use $u_{eq} \in [-1\ 1]$

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2^2 + 2u_{eq} \\ \dot{x}_2 &= x_1\end{aligned}$$

Set $\dot{\sigma}(x) = 0$

$$\dot{\sigma}(x) = \dot{x}_1 + \dot{x}_2 = -x_1 + 2x_2^2 + 2u_{eq} + x_1 = 0 \tag{6}$$

Thus $u_{eq} = -x_2^2$. Since $u_{eq} \in [-1\ 1]$ the sliding surface is between $x_2 = -1$ and $x_2 = 1$. The dynamics on the sliding surface are $\dot{\sigma}(x) = \dot{x}_1 + \dot{x}_2 = \dot{x}_1 + x_1 = 0$ which means that the dynamics are $\dot{x}_1 = -x_1$, $\dot{x}_2 = x_1 = -x_2$ which is asymptotically stable.

b. Choose Lyapunov function $V(x) = \sigma^2/2$ which gives

$$\begin{aligned}\frac{dV}{dt} &= (x_1 + x_2)(-x_1 + 2x_2^2 - 2\mathrm{sign}(x_1 + x_2) + u + x_1) && (7) \\ &= (x_1 + x_2)(2x_2^2 - 2\mathrm{sign}(x_1 + x_2) + u) && (8)\end{aligned}$$

Choose $u = -2x_2^2$. This gives

$$\frac{dV}{dt} = -2(x_1 + x_2)\mathrm{sign}(x_1 + x_2) = -2|x_1 + x_2| \le 0 \tag{9}$$

This means that we will reach the surface $\sigma(x)$ in finite time and we will stay there. The dynamics **on the sliding line** with the chosen control is

$$\begin{aligned}\dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_1 = -x_2\end{aligned}$$

which in this case is the same as without the continous feedback.

8.

a. Solve the optimal control problem

$$\begin{aligned} &\mathbf{max}\; x_2(1) && \\ &\dot{x}_1 = -u^3, && x_1(0) = 0 \\ &\dot{x}_2 = x_1 + u, && x_2(0) = 0 \end{aligned}$$

where $u(t) \geq 0$ for all $t \geq 0$. (3 p)

b. How would the solution change if $u \in [0,\ 0.1]$? (1 p)

Solution

a. The problem is normal, can put $n_0 = 1$. We have $L = 0$ and $\phi(x(1)) = -x_2(1)$

$$H = \lambda_1(-u^3) + \lambda_2(x_1 + u)$$

Hence

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = -\lambda_2 \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = 0 \end{aligned}$$

The end condition is $\lambda(1) = \phi_x$, that is $\lambda_1(1) = 0$ and $\lambda_2(1) = -1$. This gives $\lambda_2(t) \equiv -1$, $\lambda_1(t) = t - 1$. Minimization of

$$H = -(x_2 + u) - (t - 1)u^3$$

with respect to u gives

$$u^* = \sqrt{\frac{1}{3(1-t)}} > 0$$

Note: This expressions makes $\frac{\partial H}{\partial u} = 0$, but should also check that this is a minimum and also check the bound $u = 0$.

b. From the solution in **(a)** we see that the solution
$u_a^* \geq 1/3 > 0.1$.
Minimize H wrt $u \in [0,\ 0.1]$ where

$$H = \lambda_1(-u^3) + \lambda_2(x_1 + u) = \underbrace{(1-t)u^3 - u} - x_2 = \underbrace{-u(1 - (1-t)u^2)} - x_2$$

which has the solution $u_b^* = 0.1$ on the given time interval.