LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - January 16, 20, 8 am – 13 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each problem.

Preliminary grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for exercises, old exams, and solutions of these, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

Note!

In many cases the sub-problems can be solved independently of each other.

Good Luck!

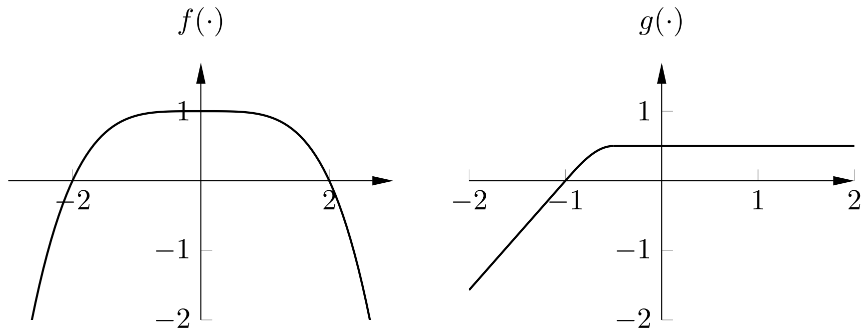


Figure 1

1. Consider the nonlinear system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \cdot f(x_1) \\
\dot{x}_2 &= g(x_2),
\end{aligned}
$$

where $f(\cdot)$ and $g(\cdot)$ are given in Figure 1.

a. Determine all equilibrium points of the system. (1.5 p)

b. Determine the characteristics of the equilibrium points that you found in **a)** (stable/unstable node, focus, saddle point, center, etc).

(1.5 p)

Solution

a. An equilibrium point must satisfy

$$
\begin{aligned}
x_2 \cdot f(x_1) &= 0 \\
g(x_2) &= 0.
\end{aligned}
$$

From the second equation it follows that $x_2 = -1$.

From the first equation it then follows that $f(x_1) = 0$, which implies $x_1 = \pm 2$.

Thus the equilibrium points are given by $(x_1, x_2) = (\pm 2, -1)$.

b. The linearization of the dynamic system around a point (x_{10}, x_{20}) is given by

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 f'(x_1) & f(x_1) \\ 0 & g'(x_2) \end{bmatrix} \Big|_{(x_1,x_2)=(x_{10},x_{20})} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
$$

Since the system matrix is triangular, its eigenvalues are given by the diagonal entries,

$$
\lambda_1 = x_{20} f'(x_{10}), \qquad \lambda_2 = g'(x_{20}).
$$

We now decide the signs of the eigenvalues to determine the stability properties of the system.

For $(x_{10}, x_{20}) = (-2, -1)$ we have from Figure 1 that $f'(-2) > 0$ and $g'(-1) > 0$, which gives that $\lambda_1 < 0$, $\lambda_2 > 0$. The equilibrium point is thus a saddle point.

For $(x_{10}, x_{20}) = (2, -1)$ we have from Figure 1 that $f'(2) < 0$ and $g'(-1) > 0$, which implies that $\lambda_1 > 0$, $\lambda_2 > 0$. The equilibrium point is thus an unstable node.

2. Consider the non-linear system

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2\,(\sin(e) - x_1) \\ e &= r - x_1, \end{aligned}$$

which can be expressed with the block diagram in Figure 2.

Prove that the dynamics from r to e are bounded-input bounded-output stable. (2.5 p)

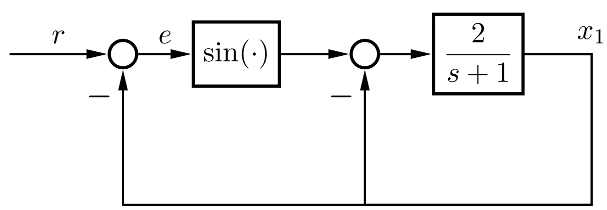


Figure 2

Solution

For convenience, let us introduce the notation $P = 2/(s+1)$. The transfer function from the output of the nonlinearity to the signal x_1 is given by

$$G(s) = \frac{P}{1+P} = \frac{2/(s+1)}{1+2/(s+1)} = \frac{2}{s+3}$$

We can thus re-draw the feedback interconnection of Figure 2 as below.

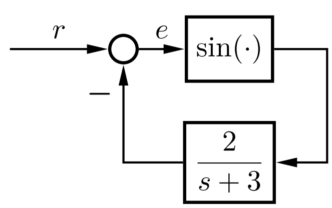


Figure 3 Simplification of the block diagram in Figure 2.

The first system in the feedback interconnection has gain $2/3$, while the other system has gain 1 (since $|\sin u| \leq |u|$). By the small gain theorem this feedback interconnection from r to e is stable. (We should also have checked that the feedback of P_1 and P_2 is internally stable).

3. Consider the system

$$\begin{aligned} \dot{x}_1 &= 3x_2 + x_1\left(4 - \frac{x_1^2}{9} - x_2^2\right) \\ \dot{x}_2 &= -\frac{x_1}{3} + x_2\left(4 - \frac{x_1^2}{9} - x_2^2\right). \end{aligned}$$

a. For which k_1 and k_2 is

$$x_1(t) = k_1 \cdot \sin(t)$$
$$x_2(t) = k_2 \cdot \cos(t)$$

a solution to the system? (1.5 p)

b. Which initial states will lead to a limit cycle?
Hint: Use the function $V(x_1, x_2) = (4 - x_1^2/9 - x_2^2)^2$ (2 p)

Solution

a. Assuming that the term $(4 - x_1^2/9 - x_2^2)$ should be zero leads to the solution $k_1 = 6, k_2 = 2$, (and $k_1 = -6, k_2 = -2$, which is an identical solution). It can also be seen that $k_1 = 0, k_2 = 0$ is a trajectory. Problem **b.** shows that these are the only solutions.
Alternatively: Plugging in the expressions leads to

$$0 = (3k_2 - k_1)\cos(t) + k_1 \sin(t)\left(4 - (k_1/3)^2)\sin^2(t) - k_2^2\cos^2(t)\right)$$
$$0 = (k_2 - k_1/3)\sin(t) + k_2\cos(t)\left(4 - (k_1/3)^2)\sin^2(t) - k_2^2\cos^2(t)\right)$$

from which it can be seen that we must have $3k_2 - k1 = 0$ and $4 - (k_1/3)^2 - k_2^2 = 0$. For example through multiplying the first equation by $k_2\cos(t)$ and subtracting $k_1\sin(t)$ times the second implies

$$0 = (3k_2 - k_1)k_2\cos(t)\sin(t) - (k_2 - k_1/3)k_1\cos(t)\sin(t), \forall t$$
$$0 = (3k_2 - k_1)k_2 - (k_2 - k_1/3)k_1$$
$$0 = 3(k_2 - k_1/3)^2 \Rightarrow k_2 = k_1/3$$

plugging it in yields

$$0 = k_1\sin(t)\left(4 - k_2^2\sin^2(t) - k_2^2\cos^2(t)\right) = k_1\sin(t)\left(4 - k_2^2\right)$$

i.e. $k_2^2 = 4$ or $k_1 = k_2 = 0$.

b. It is easy to verify that $V(x_1, x_2) = 0$ on the ellipse defined by $k_1 = 6, k_2 = 2$, and positive everywhere else. We now look at the derivative $V(x_1, x_2) = (4 - x_1^2/9 - x_2^2)^2 =: V_0(x_1, x_2)^2$ so $\dot{V} = 2V_0\dot{V}_0$ where

$$\begin{aligned}\dot{V}_0 &= \frac{\partial V_0}{\partial x} \cdot \frac{\partial x}{\partial t} \\ &= \begin{bmatrix} -\frac{2}{9}x_1 & -2x_2 \end{bmatrix} \begin{bmatrix} 3x_2 + x_1(4 - (\frac{x_1}{3})^2 - x_2^2 \\ -\frac{1}{3}x_1 + x_2(4 - (\frac{x_1}{3})^2 - x_2^2) \end{bmatrix} = \\ &= -2\left((\frac{x_1}{3})^2 + x_2^2\right)\left(4 - (\frac{x_1}{3})^2 - x_2^2\right)\end{aligned}$$

i.e.

$$\dot{V} = -2\left((\frac{x_1}{3})^2 + x_2^2\right)\left(4 - (\frac{x_1}{3})^2 - x_2^2\right)^2 \leq 0.$$

$\dot{V} = 0$ iff $x_1 = x_2 = 0$ or $V(x_1, x_2) = 0$, i.e $x_1 = 6\sin(t), x_2 = 2\cos(t)$ for some t. Using LaSalle with for example $M = \{x \mid \|x\| \geq \epsilon, V(x) \leq K\}$, with small enough ϵ and large enough K, shows that for all $x(0) \neq 0$ the solution will tend to the ellipse $V(x) = 0$.

4.a. Find a Lyapunov function and show that the origin of the following system is globally asymptotically stable.

$$\begin{aligned}\dot{x}_1 &= -x_1 + 3x_2 \\ \dot{x}_2 &= -3x_1 - x_2^3\end{aligned}$$

(2 p)

b. Design a feedback law $u = g(x_1, x_2)$ that makes the origin of the following system globally asymptotically stable

$$\begin{aligned}\dot{x}_1 &= -x_1 + 3x_2 + x_2 \arctan x_1 \\ \dot{x}_2 &= -3x_1 - x_2^3 + x_1 x_2^2 + u.\end{aligned}$$

(1.5 p)

Solution

a. Consider the Lyapunov function

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

which can be checked to be be radially unbounded, and to satisfy $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$.

We also have that

$$\begin{aligned}\dot{V} &= -x_1^2 - 3x_2x_1 + 3x_2x_1 - x_2^4 \\ &= -x_1^2 - x_2^4 \leq 0,\end{aligned}$$

with equality only for $x_1 = x_2 = 0$, this proves that the origin is globally asymptotically stable.

b. Using the same Lyqpunov function as in **a.** we get that

$$\begin{aligned}\dot{V} &= -x_1^2 - 3x_2x_1 + z_1x_2 \arctan x_1 + 3x_2x_1 - x_2^4 + x_1x_2^3 + x_2u \\ &= -x_1^2 + z_1x_2 \arctan x_1 - x_2^4 + x_1x_2^3 + x_2u.\end{aligned}$$

Choosing $u = -x_1 \arctan x_1 - x_1x_2^2$ we get $\dot{V} = -x_1^2 - x_2^4 < 0$, and global asymptotic stability.

5. When using a controller with integral action to control a process with actuator saturation, there is the well-known issue of integrator windup. Properly implemented anti-windup can remedy this problem.

The block diagram in Figure 4 shows a PI controller with anti-windup used for controlling a process with actuator saturation.

Throughout, it is assumed that $P(s)$ is stable.

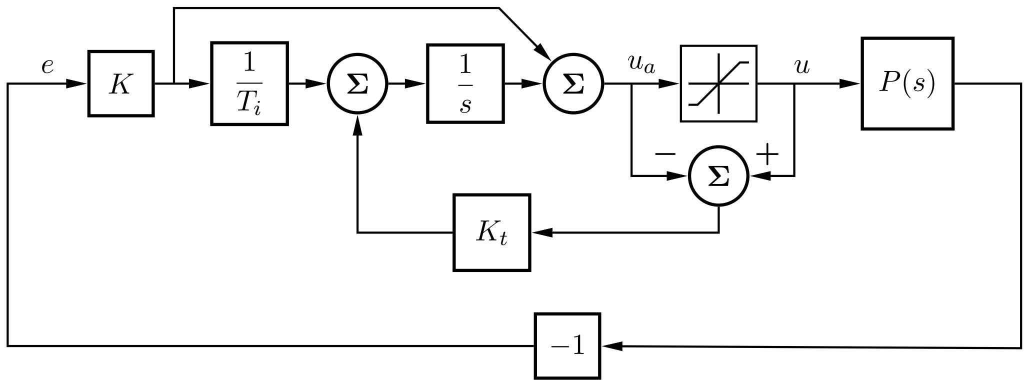


Figure 4 Feedback interconnection of a process with saturation, and a PI controller with anti-windup. Due to the actuator model in the controller, we can ignore the saturation in the process when we do the stability analysis.

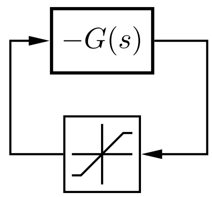


Figure 5 Transformed version of the feedback loop with anti-windup in Figure 4.

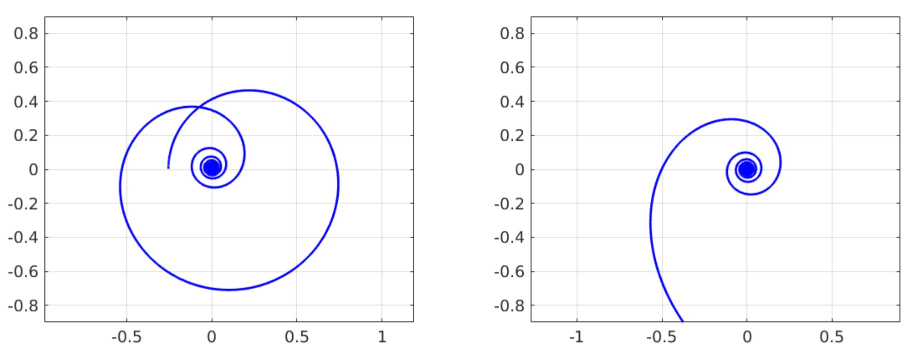
a. Isolate the saturation in Figure 4, by transforming the diagram to the form in Figure 5. What is the transfer function $G(s)$?

Use that $C(s) = K(1 + 1/(T_i s))$ to get a nicer expression. (1.5 p)

b. The Nyquist plot of the transfer function $G(s)$ in Figure 5 is shown in Figure 6a. The saturation is given by

$$\text{sat}(u) = \begin{cases} -1 & u \leq -1 \\ u & -1 \leq u \leq 1 \\ 1 & u \geq 1 \end{cases}$$

Show that the closed-loop system is asymptotically stable. (2 p)



(a) Nyquist curve for $G(s)$. (b) Nyquist curve for $P(s)C(s)$.

Figure 6

c. Consider the system *without* anti-windup. In this case the system takes the form in Figure 7, but can still be transformed to the interconnection in Figure 5 with $G(s) = P(s)C(s)$.

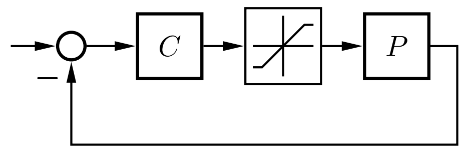


Figure 7 Feedback interconnection of a system with saturation, and a controller without anti-windup.

For a controller gain of $K = 2$, the Nyquist curve of $P(s)C(s)$ is shown in Figure 6b.

- Is the system stable?
- What would happen for larger/smaller gains K? For what range of K values does the describing function analysis predicts a limit cycle?
- Will the predicted limit cycles be stable? (2.5 p)

Solution

a. We see that

$$U_a = C \cdot E + \frac{K_t}{s}(U - U_a)$$

and using that $E = -P \cdot U$ we get

$$U_a + \frac{K_t}{s} U_a = \frac{K_t}{s} U - CPU$$

or

$$U_a = \frac{K_t/s - CP}{1 + K_t/s} U.$$

Recalling the minus sign in front of $G(s)$ in Figure 5, it follows that

$$G(s) = \frac{sC(s)P(s) - K_t}{s + K_t}.$$

b. Note that $G(s)$ has no poles in the right half plane since $P(s)$ is assumed to be stable and the pole in the origin of C is canceled by the factor s. Thus the circle criterion is applicable.

The saturation function belongs to the sector $[0, 1]$. So by the circle criterion the system is globally asymptotically stable if the transfer function $G(s)$ is to the right of the line $-1 + it$. This can immediately be verified from Figure 6a.

c. By the Nyquist criterion, the system would be stable if there was non non-linearity, so for small control errors the saturation will not be active, and we are guaranteed local asymptotic stability. However, it is not BIBO-stable since large perturbations will saturate the input and then the integrator in the controller will not be stabilized.

The describing function for the saturation nonlinearity is given in Figure 8. Drawing $-1/N/(A)$ in the complex plane thus gives a curve starting in -1 and

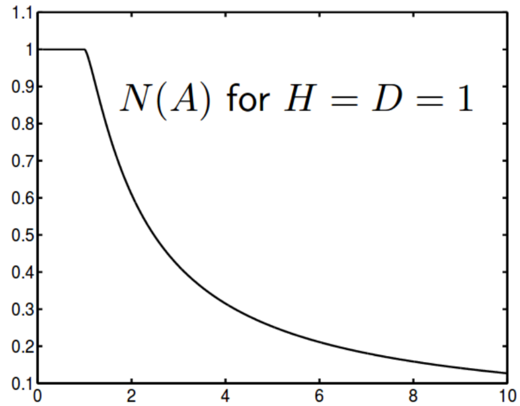


Figure 8 Describing function for a saturation.

going towards $-\infty$ on the real axis. Thus there could be limit cycles once the Nyquist curve encircles -1. This happens when the gain is twice as large as for the Nyquist curve in Figure 6b, i.e., for $K \geq 4$. No limit cycles are predicted for smaller gain $K < 4$. Note that the describing function analysis is not exact, and that there may still be limit cycles, and certainly poor control performance.

The limit cycle is stable since $-1/N(A)$ goes further to the left for increasing A.

6. Consider the discrete time system

$$x_{k+1} = f_k(x_k, u_k), \quad k = 0, 1, \ldots, N-1$$

where $f_k(x, u) = 2x + u$ and $N = 4$. We want to find the optimal state-feedback law $u_k = \mu_k(x_k)$ that minimizes the cost

$$g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k))$$

where

$$\begin{aligned} g_4(x) &= x^2 \\ g_3(x,u) &= x^2 + u^2 \\ g_2(x,u) &= x^2 + 3u^2 \\ g_1(x,u) &= x^2 + 7u^2 \\ g_0(x,u) &= x^2 + 15u^2 \end{aligned}$$

a. Use dynamic programming to compute the optimal state-feedback law with respect to the given cost function. (2.5 p)

b. Compute the optimal cost and the optimal control sequence u_0, u_1, u_2, u_3 when the initial state is $x_0 = -2$. (0.5 p)

c. Assume in addition that the control action at time $k = 0$ is limited to $|u_0| \leq 1$. Compute the cost-to-go function $V_0(x_0)$. (0.5 p)

Solution

a. Define

$$V_k(x_k) = g_N(x_N) + \sum_{j=k}^{N-1} g_j(x_j, \mu_j^*(x_j))$$

where $\mu^* = \{\mu_0^*, \mu_1^*, \ldots, \mu_{N-1}^*\}$ is an optimal control policy. Following the lecture notes, we have

$$\begin{aligned}
V_4(x) &= g_4(x) = x^2 \\
V_3(x) &= \min_u \left[g_3(x,u) + V_4(2x+u)\right] \\
&= \min_u \left[x^2 + u^2 + (2x+u)^2\right] = 3x^2 \quad \text{(the minimum is attained for } u = -x\text{)} \\
V_2(x) &= \min_u \left[g_2(x,u) + V_3(2x+u)\right] \\
&= \min_u \left[x^2 + 3u^2 + 3(2x+u)^2\right] = 7x^2 \quad \text{(the minimum is attained for } u = -x\text{)} \\
V_1(x) &= \min_u \left[g_1(x,u) + V_2(2x+u)\right] \\
&= \min_u \left[x^2 + 7u^2 + 7(2x+u)^2\right] = 15x^2 \quad \text{(the minimum is attained for } u = -x\text{)} \\
V_0(x) &= \min_u \left[g_0(x,u) + V_1(2x+u)\right] \\
&= \min_u \left[x^2 + 15u^2 + 15(2x+u)^2\right] = 31x^2 \quad \text{(the minimum is attained for } u = -x\text{)}
\end{aligned}$$

The minimal values are attained by the control law $u_k = -x_k$ for $k = 0, 1, 2, 3$.

b. The optimal control law generates the sequence

$$u_3 = u_2 = u_1 = u_0 = 2.$$

and the optimal value $V_0(x_0) = 31x_0^2 = 124$.

c. The control constraint $|u_0| \leq 1$ changes only the last step in the dynamic programming sequence:

$$\begin{aligned}
V_0(x) &= \min_{|u|\leq 1} \left[x^2 + 15u^2 + 15(2x+u)^2\right] \\
&= \min_{|u|\leq 1} \left[31x^2 + 30(x+u)^2\right] \\
&= \begin{cases} 31x^2 & \text{if } |x| \leq 1 \\ 31x^2 + 30(x - \operatorname{sgn}(x))^2 & \text{otherwise} \end{cases}
\end{aligned}$$

7. Solve the optimal control problem

$$\begin{aligned}
\text{maximize} \quad & x_2(1) \\
\text{subject to} \quad & \dot{x}_1 = -u^3, \qquad x_1(0) = 0 \\
& \dot{x}_2 = x_1 + u, \quad x_2(0) = 0 \\
& u(t) \geq 0.
\end{aligned}$$

(3 p)

Solution

The problem is normal, so $n_0 = 1$. We have $L = 0$ and $\phi(x(1)) = -x_2(1)$ (the minus sign enters since it is a maximization problem).

The Hamiltonian is given by

$$H = \lambda_1(-u^3) + \lambda_2(x_1 + u).$$

Hence

$$\begin{aligned}\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = -\lambda_2 \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = 0\end{aligned}$$

The end condition is $\lambda(1) = \phi_x$, that is $\lambda_1(1) = 0$ and $\lambda_2(1) = -1$. This gives $\lambda_2(t) \equiv -1$, $\lambda_1(t) = t - 1$. Minimization of

$$H = -(x_2 + u) - (t - 1)u^3$$

with respect to u gives

$$u^* = \sqrt{\frac{1}{3(1-t)}}$$

Note: This expressions makes $\partial H/\partial u = 0$, but you should also check that this is a minimum.