Last week

- State Space Realizations (pp 139-150)
- $G(s)$, denominator and numerator, poles and zeros
- Change of coordinates, diagonal and controllable form
- State-feedback
- Observers
- Feedback from estimated states
- Integral action by disturbance model
Controllability – Existence of control signal
- Which state directions can be controlled?
Observability – Determine state
- Which state directions can not be seen?
Kalman’s decomposition theorem
Cancelled dynamics \(\iff\) lack of controllability or observability
How should **controllability** be defined?

Some (not used) alternatives:

By proper choice of control signal \( u \)

- any state \( x_0 \) can be made an equilibrium
- any state trajectory \( x(t) \) can be obtained
- any output trajectory \( y(t) \) can be obtained

The most fruitful definition has instead turned out to be the following
The state equation

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \]

is called *controllable* if for any \( x_0 \) and \( T > 0 \), there exists \( u(t) \) such that \( x(T) = 0 \) (“Controllable to origin”)

Question: Is this equivalent to the following definition:

“for \( x_0 = 0 \) and any \( x_1 \) and \( T > 0 \), there exists \( u(t) \) such that \( x(T) = x_1 \)” (“Controllable from origin”)

The audience is thinking!

*Hint:* \( x(T) = e^{AT}x_0 + \int_0^T e^{A(T-t)}Bu(t)\,dt \)
The matrix

\[ W(T) = \int_0^T e^{-At} BB^T e^{-A^T t} dt \]

is called the controllability Gramian.

Note that it is positive semidefinite, \( W(T) \geq 0 \)

The main controllability result is the following
Theorem Controllability Test

The following conditions are equivalent:

(i) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable.

(ii) $\text{rank } [B\ AB\ A^2B\ \ldots\ A^{n-1}B] = n$. 

(iii) $W(T)$ is invertible for any $T > 0$ 

(iv) For any $\lambda \in \mathbb{C}$ we have $\text{rank } [A - \lambda I\ B] = n$

We will prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$

The condition (iv), not proved here, is called the PBH test (Popov-Belevitch-Hautus).
Analysing controllability

The system is by definition controllable iff we for any $x_0$ and $T$ can find control signal $u(t), t \in [0, T]$ that solves (see hint some slides above)

$$-x_0 = \int_0^T e^{-At} Bu(t)dt \quad (\star)$$

Cayley-Hamilton’s theorem (google it) says that $A^k$ for $k \geq n$ can be written as a linear combination of $I, A, A^2, \ldots A^{n-1}$, so

$$e^{-At} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A^k = \sum_{k=0}^{n-1} f_k(t) A^k, \quad \text{(for some } f_k(t))$$

Therefore the condition $(\star)$ can be written

$$-x_0 = [B \ AB \ A^2 B \ldots A^{n-1} B] F(u), \quad (\star\star)$$

for some vector $F(u)$ with elements $F_k(u) = \int_0^T f_k(t)u(t)dt$
Proof by contradiction: Assume (ii) does not hold, i.e. the controllability matrix does not have full rank.

This means there is a vector, let's call it $-x_0$, that is not in the column span of

$$[B \ AB \ A^2B \ \ldots \ A^{n-1}B]$$

This contradicts (⋆⋆), so (i) does not hold.
Proof of \(( ii \) \( \Rightarrow \) \( iii \))

Assume \(( iii \) does not hold. Then there is a \( p \neq 0 \) so \( W(T)p = 0 \).

\[
0 = p^T W(T)p = \int_0^T \left( p^T e^{-At} B \right) \left( B^T e^{-A^T t} p \right) dt
\]

Therefore

\[
p^T e^{-At} B = 0, \forall t.
\]

Derivating this \( k \) times and setting \( t = 0 \) gives \( p^T A^k B = 0 \).

Hence we have

\[
p^T [B \ AB \ A^2 B \ldots A^{n-1} B] = 0.
\]

Therefore \(( ii \) does not hold.
(iii) $\Rightarrow$ (i) Explicit construction of $u(t)$

If $W(T)$ is invertible, then for any initial state $x_0$, the control signal

$$u(t) = -B^T e^{-A^T t} (W(T))^{-1} x_0$$

gives $x(T) = 0$ (check that $(\star)$ some slides before is satisfied!). Hence the system is controllable.
Another interpretation of $W(T)$

One can prove (using techniques from next lecture) that the minimal (squared) control energy, defined by $\|u\|^2 := \int_0^T |u|^2 dt$, needed to move from $x(0) = x_0$ to $x(T) = 0$ equals

$$x_0^T (W(T))^{-1} x_0$$

Gives nice formula for which state directions are costly to control.

$W(T)$ large in some direction means easy to control in that direction.
Which trailer is controllable?
\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} = \begin{bmatrix}
-2 & 0 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix} + \begin{bmatrix}
2 \\
0
\end{bmatrix} u
\]
\[ \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \]
\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix} + \begin{bmatrix}
1 \\
1
\end{bmatrix} u
\]
The system

\[
\begin{cases}
\frac{dx}{dt} = Ax, & x(0) = x_0 \\
y = Cx
\end{cases}
\]

is called observable if \( x_0 \) can be uniquely determined from \( y_{[0,T]} \) (for any \( T > 0 \))

This is the same as saying that the only \( x_0 \) for which \( y(t) = 0 \) for all \( t \)

is the trivial case \( x_0 = 0 \)

WHY? The audience is thinking!
Which trailer is observable? If $y = \theta_1$? If $y = \theta_2$?
Theorem - Observability Criteria

The following are equivalent

(i) The system \( \dot{x} = Ax, \ y = Cx \) is observable

\[
\begin{pmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{pmatrix}
\]

(ii) \( \text{rank} \left( \begin{pmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{pmatrix} \right) = n \)

(iii) \( \widetilde{W}(T) \) is invertible for any \( T > 0 \)

(iv) For any \( \lambda \in \mathbb{C} \) we have \( \text{rank} \left( \begin{pmatrix}
A - \lambda I \\
C
\end{pmatrix} \right) = n \)

Here the observability Gramian \( \widetilde{W}(T) \) is defined as

\[
\widetilde{W}(T) = \int_0^T e^{A^T t} C^T C e^{A t} dt
\]
Proof that \((i) \iff (ii)\)

If \((i)\) does not hold, then there is a quiet state \(x_0 \neq 0\) so that

\[
y(t) = C e^{At} x_0 = 0, \quad \forall t
\]

Derivating this \(k\) times and setting \(t = 0\) we get \(CA^k x_0 = 0\). This shows \((ii)\) doesn’t hold.

On the other hand, if \((ii)\) does not hold then a nonzero \(x_0\) can be found so \(CA^k x_0 = 0\) for \(k = 0, \ldots n - 1\). By Cayley-Hamilton this means \(CA^k x_0 = 0\) also for \(k \geq n\), so by power expansion of \(e^{At}\)

\[
y(t) = C e^{At} x_0 = 0, \quad \forall t,
\]

which says that \((i)\) does not hold.
This follows easily by substituting \((A, B)\) with \((A^T, C^T)\) in (ii) and (iii) in the controllability theorem earlier.

This illustrates a so called **duality** between the two theorems.
Maybe you didn’t like the earlier proof of \((ii) \Rightarrow (i)\) that used derivation of \(y(t)\). It is hard to implement in practice. If there e.g. is measurement noise on \(y(t)\) we would like a better way of determining \(x_0\).

The dual result of the construction of the energy-optimal \(u(t)\) in the controllability theorem is the following calculation:

Since \(y(t) = C e^{At} x_0\) we have that

\[
\int_0^T e^{A^T t} C^T y(t) dt = \tilde{W}(T)x_0
\]

When \(\tilde{W}(t)\) is invertible we can hence find \(x_0\) by

\[
x_0 = (\tilde{W}(T))^{-1} \int_0^T e^{A^T t} C^T y(t) dt.
\]

This way of determining \(x_0\) is actually optimally robust against measurement noise, in a sense described in Lecture 6.
\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} =
\begin{bmatrix}
-2 & 0 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]
\[
y =
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]
\[
y =
\begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]
\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} =
\begin{bmatrix}
-2 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]
Theorem:
If the system is noncontrollable, say \( \text{rank}(C) = q < n \), then there is a state transformation \( x = Vz \) so that in the new state coordinates
\[
AV = V \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \quad \text{and} \quad B = V \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix},
\]
\((\tilde{A}_{11}, \tilde{B}_1)\) controllable subsystem, \( q \times q \)
Observability – state transformation

Theorem:
If the system is non-observable, say \(\text{rank}(\mathcal{O}) = q < n\), then there is a state transformation so that in the new state coordinates

\[
\begin{align*}
AV &= V \begin{pmatrix}
\tilde{A}_{11} & 0 \\
\tilde{A}_{21} & \tilde{A}_{22}
\end{pmatrix} \\
CV &= \begin{pmatrix}
\tilde{C}_1 & 0
\end{pmatrix},
\end{align*}
\]

\((\tilde{A}_{11}, \tilde{C}_1)\) observable subsystem, \(q \times q\)
Kalman’s decomposition theorem

With a state transformation that splits the controllable subspace (and its complement) into nonobservable subspace and complement we get the system on a nice form

\[
\frac{dx}{dt} = \begin{pmatrix}
A_{11} & 0 & A_{13} & 0 \\
A_{21} & A_{22} & A_{23} & A_{24} \\
0 & 0 & A_{33} & 0 \\
0 & 0 & A_{43} & A_{44}
\end{pmatrix} x + \begin{pmatrix}
B_1 \\
B_2 \\
0 \\
0
\end{pmatrix} u
\]

\[
y = \begin{pmatrix}
C_1 & 0 & C_2 & 0
\end{pmatrix} x
\]

\[
G(s) = C_1(sI - A_{11})^{-1}B_1
\]

Illustrates what subparts of the system that influences the input-output behavior
The audience is thinking: What blocks in this figure corresponds to parts 1, 2, 3, 4 on the previous slide?
Kalman’s decomposition theorem

If no common eigenvalues between any two blocks on the diagonal, then corresponding off-diagonal blocks can be eliminated by changed choice of the complementing spaces. Simplifies picture further
\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 2 & -2
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix} +
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} u,
\quad
y =
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix}
\]

What does the decomposition theorem say when \( y = \theta_2 \)? What block is then missing?
Trailer 4 after coordinate change

\[
\begin{bmatrix}
\dot{\theta}_2 \\
\dot{\theta}_3 \\
\dot{\theta}_1 - \dot{\theta}_2
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 \\
2 & -2 & 0 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\theta_2 \\
\theta_3 \\
\theta_1 - \theta_2
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_2 \\
\theta_3 \\
\theta_1 - \theta_2
\end{bmatrix}
\]

controllable and observable subsystem: $\theta_2$
Zeros and state feedback

Remember: State-feedback does not change zeros.

Choose state feedback $L$ that gives a pole in $\lambda$.

If the mode $x_0 e^{\lambda t}$ now becomes non-observable

$$\begin{pmatrix} A - BL - \lambda I \\ C \end{pmatrix} x_0 = 0$$

then actually $\lambda$ was a zero to the system:

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = 0$$

Corresponds to cancellation of the factor $s - \lambda$ in

$$G(s) = C(sI - A + BL)^{-1}Bl_r$$
Given two systems \( n_i(s)/d_i(s) = c_i(sI - A_i)^{-1}b_i, \ i = 1, 2 \)

Then the series connection \( \frac{n_2(s)}{d_2(s)}\frac{n_1(s)}{d_1(s)} \) is

- uncontrollable \( \iff \) there is \( \lambda \) so \( n_1(\lambda) = d_2(\lambda) = 0 \)
- unobservable \( \iff \) there is \( z \) so \( n_2(\lambda) = d_1(\lambda) = 0 \)

Proof:

Controllable, check when \( \text{rank} \begin{bmatrix} \lambda I - A_1 & 0 & b_1 \\ -b_2c_1 & \lambda I - A_2 & 0 \end{bmatrix} \leq n \)

Observable, check when \( \text{rank} \begin{bmatrix} \lambda I - A_1 & 0 \\ -b_2c_1 & \lambda I - A_2 \end{bmatrix} \leq n \)
Cancellation in series connections

Example

\[ Y(s) = \frac{s + 3}{s - 1} \cdot \frac{s - 1}{s + 2} U(s) \]

Loss of controllability of an unstable mode. Bad.

Example

\[ Y(s) = \frac{s - 1}{s + 2} \cdot \frac{s + 3}{s - 1} U(s) \]

Loss of observability of an unstable mode. Also bad.
Summary

- Controllability - criteria
- Observability - criteria
- Kalman’s decomposition
- Cancelled dynamics \(\iff\) lack of controllability or observability