### Last Week

- Laplace transform - single vs double sided
- Initial and Final Value Theorem

### Initial and Final Value Theorem

**Initial Value Theorem** Suppose that \( f \) is causal and that the Laplace transform \( F(s) \) is rational and strictly proper. Then
\[
\lim_{t \to +\infty} f(t) = \lim_{s \to 0} sF(s)
\]

**Final Value Theorem** Suppose that \( f \) is causal with rational Laplace transform \( F(s) \). If all poles of \( sF(s) \) have negative real part, then
\[
\lim_{t \to +\infty} f(t) = \lim_{s \to +\infty} sF(s)
\]

### Lecture 2

- (Cauchy’s) Argument Principle
- Nyquist criterion
- Example: Trailer
- Example: Feedback with time delay
- Bode’s relations between gain and phase

### Argument variation

Let \( \Gamma \) be a simple closed curve in the complex plane surrounding the domain \( D \).

The change in the argument for the complex function \( F(s) \) when \( s \) follows the boundary to \( D \) (i.e., follows \( \Gamma \)) in a counter-clockwise (CCW) direction, is called the argument variation of \( F \) along \( \Gamma \) and is denoted \( \Delta \Gamma \text{ arg } F \):
\[
\Delta \Gamma \text{ arg } F := \int_{\Gamma} \left( \frac{d}{ds} \arg F(s) \right) ds
\]

### (Cauchy’s) argument principle

Suppose that \( F(s) \) is analytic in a neighborhood of \( D \) except for a finite number of poles in \( D \). Then
\[
\frac{1}{2\pi} \Delta \Gamma \text{ arg } F = N_F - P_F
\]

where \( N_F \) is the number of zeros and \( P_F \) the number of poles of \( F \) in \( D \).

### Proof of the Argument Principle

The argument function is the imaginary part of the complex logarithm, so
\[
\Delta \Gamma \text{ arg } F = \int_{\Gamma} \left( \frac{d}{ds} \arg F(s) \right) ds
\]

\[
= \lim_{t \to +\infty} F(t) = \lim_{s \to +\infty} sF(s)
\]

\( F' / F \) is singular exactly in the poles and zeros of \( F \).

### Proof cont’d

\[
F(s) = \frac{(s - z_1) \cdots (s - z_{N_F})}{(s - p_1) \cdots (s - p_{P_F})} G(s)
\]

where \( G \) has no poles and zeros in \( D \). Then
\[
\log F(s) = \sum_{j=1}^{N_F} \log(s - z_j) - \sum_{j=1}^{P_F} \log(s - p_j) + \log G(s)
\]

Derivation and integration gives
\[
\lim_{t \to +\infty} \int f(t) \frac{d}{ds} \log F(s) ds = \lim_{s \to +\infty} \int \left( \sum_{j=1}^{N_F} \frac{1}{s - z_j} - \sum_{j=1}^{P_F} \frac{1}{s - p_j} + \frac{G'(s)}{G(s)} \right) ds
\]

\[
= N_F - P_F
\]

### Lecture 2

- (Cauchy’s) Argument Principle
- Nyquist criterion
- Example: Trailer
- Example: Feedback with time delay
- Bode’s relations between gain and phase
**Nyquist Criterion**

Regler AK: If $L(s)$ is stable, then the closed loop system $(1 + L(s))^{-1}$ is also stable if and only if the Nyquist curve $L(i\omega)$ does not encircle $-1$.

More general: The difference of the number of unstable poles to $(1 + L(s))^{-1}$ and the number of unstable poles of $L(s)$ equals the number of clockwise encirclements of the point $-1$.

![Nyquist Diagram](image)

**Proof of the Nyquist criterion**

Apply the argument principle on $F(s) = 1 + L(s)$, where $D$ as in picture, and radius large enough to contain all poles and zeros in the RHPL.

Then

- $P_F$ = number of unstable poles to $1 + L(s)$ = $P_{open}$
- $N_F$ = number of unstable poles to $(1 + L(s))^{-1}$ = $P_{closed}$
- $\frac{1}{2\pi} \Delta \arg F$ = number of CCW encirclements of 0 by $F(s)$ when $s$ moves around boundary of $D$ CCW

$P_{closed} - P_{open}$ = nr of clockwise encirclements around $-1$ of $L(i\omega)$

**Example: Trailer**

When trailer moves forward with speed $v = 1$:

$$Y(s) = \frac{1}{s + 2(s+1)} U(s)$$

**Example: Trailer moving forward with P-control**

P-control: $U(s) = -kY(s)$. Gives $L = kG$.

$s = \text{tf}(s^*)$
$G = \frac{1}{(s+2)(s+1)}$

nyquist(G)

Stable if $L(i\omega) = k \frac{1}{(s+2)(s+1)}$ does not encircle $-1$.

True for all $k > 0$ (and some $k < 0$)

**Example: Trailer moving backwards with P-control**

Let's try this PD-controller: $U(s) = -k(1 + s)Y(s)$.

Stable if $L(i\omega) = k \frac{1+i\omega}{(s+2)(s+1)}$ encircles $-1$ two times counter-clockwise.

True when $k > 3$. PD-control works

**Example: Trailer moving backwards with PD-control**

Now $G(s) = \frac{1}{(s-2)(s-1)}$

G = 1/((s-2)*(s-1))

nyquist(G)

When does $L(i\omega) = k \frac{1}{(s-2)(s-1)}$ encircle $-1$ two times counter-clockwise?

Never. So P-control can not be used.

**Lecture 2**

- (Cauchy's) Argument Principle
- Nyquist criterion
- Example: Trailer
- Example: Feedback with time delay
- Bode’s relations between gain and phase
Disturbance rejection

If the system is stable,
\[ \dot{y}(t) = y(t) - 2y(t - 0.5) \]

stable?
This can be viewed as a feedback system
\[ \dot{y}(t) = y(t) + u(t) \]
\[ u(t) = -2y(t - 0.5) \]

Can use Nyquist criterion with \( L = P(s)C(s) = \frac{2e^{-0.5s}}{s-1} \)

Stable, since \( L(i\omega) = \frac{2e^{-i0.5\omega}}{i\omega-1} \) encircles -1 one time counter clockwise.

Lecture 2

- (Cauchy’s) Argument Principle
- Nyquist criterion
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Design tradeoffs

A control system should typically have high gain \(|P(i\omega)C(i\omega)|\) at low frequencies to reduce impact of disturbances and to follow the reference signal \( r \), but low gain at high frequencies to avoid stability problems and the effect of measurement noise

Example: System with time delay
\[ \dot{y}(t) = y(t) - 2y(t - 0.5) \]

Stable, since \( L(i\omega) = \frac{2e^{-i0.5\omega}}{i\omega-1} \) encircles -1 one time counter clockwise.

Bode’s relations — Approximative version

If \( G(s) \) is stable and has no zeros in the RHPL and no time delay then
\[ \arg G(i\omega_0) = \frac{\pi}{2} \left. \frac{d \log |G(i\omega)|}{d \log \omega} \right|_{\omega=\omega_0} \]

If there are zeros in the RHPL or time delay the phase will be smaller

Conclusion: The slope of the amplitude determines the phase.
Phase -180 degree corresponds to slope -2 (with log-log scales)
At the cut off frequency (where the amplitude equals one) the slope needs to be > -2 (around -1.5 is recommended). Otherwise the Nyquist curve will go the wrong way around -1
Can not reduce loop gain too fast.

Bode’s relation(s) — Exact version

If \( G(s) \) is stable and minimum phase (no zeros in RHPL or time delays) then
\[ \arg G(i\omega_0) = \frac{1}{\pi} \int_0^\infty \frac{d \log |G(i\omega)|}{d \log \omega} \log \frac{\omega + \omega_0}{\omega - \omega_0} \, d\omega \]

Bode’s relation – Proof

- We first show
  \[ \arg G(i\omega_0) = \frac{2\omega_0}{\pi} \int_0^\infty \frac{\log |G(i\omega)| - \log |G(i\omega_0)|}{\omega^2 - \omega_0^2} \, d\omega \]
- Changes of variables and partial integration give
  \[ \arg G(i\omega_0) = \frac{1}{\pi} \int_0^\infty \frac{d \log |G(i\omega)|}{d \log \omega} \log \frac{\omega + \omega_0}{\omega - \omega_0} \, d\omega \]

Lecture 2

- (Cauchy’s) Argument Principle
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Bode Magnitude Diagram
Frequency (rad/sec)
Magnitude (dB)

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<th>10^-2</th>
<th>10^-1</th>
<th>10^0</th>
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Disturbance rejection

Robustness

PC

Bode’s relations between gain and phase

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<th>y</th>
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<td>10^2</td>
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</tbody>
</table>
Bode's relation – Proof cont’d

Let $C$ be the depicted curve, then
\[
\int_C \log G(s) - \log |G(i\omega_0)| - \log |G(i\omega)| ds = 0
\]
since the function is analytic on and inside $C$

Integral over $C$ satisfies:
\[
\int_C = \int_{C_R} + \int_{-\infty}^{\infty} + \int_{CR} = 0
\]

Bode's relation – Proof cont’d

To prove the second claim, we change variable $\omega = e^x$:
\[
\int_{-\infty}^{\infty} \log |G(i e^x)| - \log |G(i\omega_0)| e^x dx = \int_{-\infty}^{\infty} \log |G(i e^x)| - \log |G(i\omega_0)| \frac{1}{e^x - e^{-x}} dx
\]

Define
\[
\phi(x) = \log e^x + \frac{1}{e^x - 1} \quad \text{with} \quad \frac{d}{dx} \phi(x) = -\frac{2}{e^x - e^{-x}}
\]
then
\[
\frac{d}{dx} \phi(x - \log \omega_0) = -\frac{2}{e^{x - \log \omega_0}} = \frac{2}{e^x / \omega_0 - e^{-x} / \omega_0} = \frac{2\omega_0}{e^x - \omega_0 e^{-x}}
\]

Bode's relation – Proof cont’d

Partial integration gives
\[
\frac{2\omega_0}{\pi} \int_{-\infty}^{\infty} \log |G(i e^x)| - \log |G(i\omega_0)| \frac{1}{e^x - \omega_0 e^{-x}} dx = -\frac{1}{\pi} \int_{-\infty}^{\infty} \log |G(i e^x)| - \log |G(i\omega_0)| \frac{d}{dx} \phi(x - \log \omega_0) dx
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{dx} \log |G(i e^x)| \phi(x - \log \omega_0) dx
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{dx} \log |G(i e^x)| \phi(x - \log \omega_0) \big|_{x=-\infty}^{x=\infty}
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{dx} \log |G(i e^x)| \phi(x - \log \omega_0) dx
\]

Changing variables back, $x = \log \omega$, gives:
\[
\arg G(i\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{d \log \omega} \log |G(i e^{\log \omega})| e^{\log \omega - \log \omega_0} + 1 \frac{d}{d \log \omega} \log \omega - \log \omega_0 - 1 \frac{d}{d \log \omega} \log \omega
\]
which is readily rewritten to Bode's relation

Bode's relation – Proof cont’d

Integral on $C_R$: $\int_{C_R} = 0$ as $R \to \infty$ (proper)

Integral on (both) $C_r \quad (r \to 0)$:
\[
\int_C \frac{\log |G(s) - \log |G(i\omega_0)|}{s^2 + \omega_0^2} ds = \int_C \frac{\log |G(s) - \log |G(i\omega_0)|}{(s - i\omega_0)(s + i\omega_0)} ds
\]
\[
= \int_C \frac{\log |G(s)| e^{\arg G(s)/|G(i\omega_0)|}}{s - i\omega_0)ds} ds
\]
\[
= \frac{1}{2\omega_0} \int_C \frac{\log |G(i\omega_0)|}{s - i\omega_0 ds}
\]
\[
= \frac{1}{2\omega_0} \int_C \frac{1}{s - i\omega_0 ds} = \frac{i \arg G(i\omega_0)}{2\omega_0}
\]

Therefore, when $R \to \infty$ and $r \to 0$:
\[
\frac{i \arg G(i\omega_0)}{\omega_0} = \int_{-\infty}^{\infty} \log |G(i\omega)| - \log |G(i\omega_0)| \frac{1}{\omega_0^2 - \omega^2} d\omega
\]

Bode's relation – Proof cont’d

Rewrite from previous slide:
\[
\arg G(i\omega_0) = \frac{-\omega_0}{\pi} \int_{-\infty}^{\infty} \log |G(i\omega)| - \log |G(i\omega_0)| \frac{1}{\omega^2 - \omega_0^2} d\omega
\]

Bode's relation – Proof cont’d

Since $\log |G(i\omega)| = \log |G(i\omega_0)| + i \arg G(i\omega_0)$ and $\arg G(i\omega)$ is odd:
\[
\arg G(i\omega) = \frac{2\omega_0}{\pi} \int_{-\infty}^{\infty} \log |G(i\omega)| - \log |G(i\omega_0)| \frac{1}{\omega^2 - \omega_0^2} d\omega
\]
which shows the first claim

Bode's relation – Proof cont’d

Hint to problem 1c

If one first determines $Y(s)$ one can then have use of the fact that for any complex number $v$ we have the identity
\[
(sI - A)^{-1}(s - v)^{-1} = -(sI - A)^{-1}(sI - A)^{-1} + (sI - A)^{-1}(s - v)^{-1}.
\]
(If you use this identity, you should prove it!) Apply with $v = i\omega$ and $v = -i\omega$, combine the results and do inverse laplace.

Also remember that $\text{Im}(z) = (z - \bar{z})/(2i)$ and $\text{Im}(e^{i\omega t}) = (e^{i\omega t} + e^{-i\omega t})/2$.