Lecture 1

- Course contents
- Practical stuff - book - today pp. 71-101
- Math background
- Laplace transform
- Transient and initial states
- AK background - frequency curves AK 27-39

Course content

Lec1	Basic system theory
Lec2	Argument variation principle, Nyquist theorem, Bode's relations
Lec3	Stability, Robustness, Sensitivity Function
	w7 HANDIN 1: Laplace transform and Frequency plots.
Lec4	State coordinate change, zeros, state feedback, observers
Lec5	Controllability and Observability, Kalman's decomposition theorem
Lec6	Linear mappings and least squares problems
	w10: HANDIN 2: State representations

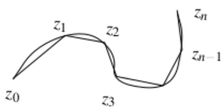
Presentations HANDIN 1: TBD
no presentations HANDIN2

Komplex kurvintegral

Vad är en komplex kurvintegral?

Antag att $f(z)$ är en komplex funktion och att C är en kurva i det komplexa talplanet. Man kan då beräkna den *komplexa kurvintegralen* av f över C så här; gå genom kurvan under ett intervall $a \le t \le b$, dvs $z = z(t)$ genomlöper kurvan. Sampla intervallet som $a = t_0 \le t_1 \le \cdots \le t_n = b$. Det ger punkterna $z_k = z(t_k)$ på kurvan. Då är

$$\int_C f(z)\,dz \approx \sum_k f(z_k)\Delta z_k = $$
$$= \sum_k f(z_k)\frac{\Delta z_k}{\Delta t_k}\Delta t_k \approx \int_a^b f(z)\frac{dz}{dt}\,dt.$$



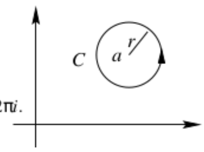
(Från: Spannes blixtkurs i komplex integration)

Exempel

Beräkna integralen av $f(z) = 1/(z-a)$ över $C = \{z : |z-a| = r\}$ genomlöpt ett varv i positivt led

Kurvan ges av $z = a + re^{it}$, $0 \le t \le 2\pi$.
Detta ger $\frac{dz}{dt} = rie^{it}$ och alltså är

$$\int_C \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$



(Från: Spannes blixtkurs i komplex integration)

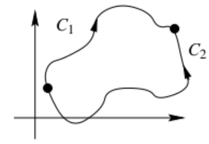
Cauchys integralsats

- Antag f analytisk på och innanför sluten kurva C
- Komplex integral ($f = u + iv$, $z = x + iy$, $dz = dx + idy$):
$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy$$
- Greens formel (D är C och allt innanför C):
$$\int_C udx - vdy = \int\int_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$
$$\int_C vdx + udy = \int\int_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$
- Cauchy Riemanns ekvationer
$$\frac{du}{dy} = -\frac{dv}{dx} \qquad \frac{du}{dx} = \frac{dv}{dy}$$
- Cauchys integralsats:
$$\int_C f(z)dz = 0$$

Integraler blir lika

Antag nu att två kurvor C_1 och C_2 har samma begynnelsepunkt och samma slutpunkt. Om $f(z)$ är analytisk *överallt* mellan kurvorna, så är

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$



Detta följer av att kurvan $C = C_1 - C_2$ (först C_1, sedan C_2 baklänges) är sluten och $f(z)$ är analytisk överallt innanför den. Alltså är

$$\int_C = \int_{C_1} - \int_{C_2} = 0$$

(Från: Spannes blixtkurs i komplex integration)

Cauchys integralformel

Om

- C är en sluten positivt orienterad kurva (som inte skär över sig själv)
- $f(z)$ är analytisk innanför (och på) C
- a är en punkt innanför C

så är

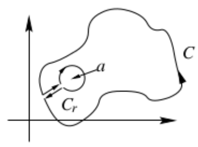
$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.$$

Man kan alltså rekonstruera f innanför kurvan med hjälp endast av dess värden på kurvan.

(Från: Spannes blixtkurs i komplex integration)

Förenklad härledning

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_r} \frac{f(z)}{z-a}\,dz \approx$$
$$\approx f(a)\int_{C_r} \frac{1}{z-a}\,dz = f(a)2\pi i.$$

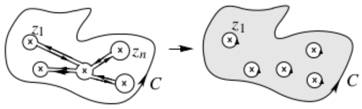


Först flyttar vi integrationsvägen till en liten cirkel kring a, sedan använder vi att $f(z) \approx f(a)$ om $z \approx a$.

(Från: Spannes blixtkurs i komplex integration)

Residuekalkyl

Idén för metoden är samma som i beviset av Cauchys integralformel. Man flyttar integrationskurvan utan att gå över singulariteter (punkter där f ej är analytisk). Då ändras inte integralens värde. Som framgår av figuren kan man ersätta det ursprungliga C-et med små cirklar, en kring varje pol.



Detta kan skrivas

$$\int_C = \int_{C_1} + \int_{C_2} + \cdots + \int_{C_n}$$

(Från: Spannes blixtkurs i komplex integration)

Residuekalkyl

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi i}\int_{C(a,r)} f(z)\,dz$$

Förutsättningarna är

- $C(a,r)$ är en (liten) cirkel med radie r och medelpunkt a, genomlöpt ett varv i positiv led
- funktionen $f(z)$ är analytisk överallt innanför $C(a,r)$ men ej (nödvändigtvis) i $z = a$.

Då beror värdet av integralen inte på r. (Faktorn $2\pi i$ gör det lättare att beräkna residyerna.) Resultatet är *Residysatsen:*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

(Från: Spannes blixtkurs i komplex integration)

Laplace transform

- Double vs single sided Laplace
- Strip of definition. Different for different signals
- Transfer functions. How do we handle different strips of definition?
- Use one sided transforms + analytic continuation
- Makes it possible to also analyse unstable causal systems

Laplace transform - definition - convergence

Double-sided (notation $\mathcal{L}_{II}$):

- Consider time functions $f(t)$, $-\infty < t < \infty$

$$F(s) = (\mathcal{L}_{II} f)(s) = \int_{-\infty}^{\infty} e^{-st} f(t)dt$$

- Convergence in strip

$$\Omega = \{s : \alpha < Re\ s < \beta\}, \quad \text{with } F(s) \text{ analytic in } \Omega$$

needs

$$e^{-\alpha t} f(t) \to 0, \quad t \to \infty, \quad \text{and} \quad e^{-\beta t} f(t) \to 0, \quad t \to -\infty$$

- Example: $\alpha < 0$ and $\beta > 0$ requires exponential convergence for both $t \to \infty$ and $t \to -\infty$.

Laplace transform - definition - convergence

Single-sided (notation $\mathcal{L}_I$):

- Consider $f(t)$, $0 \le t < \infty$

$$F(s) = (\mathcal{L}_I f)(s) = \int_0^{\infty} e^{-st} f(t)dt$$

- Convergence in half plane

$$\Omega = \{s :\ \alpha < Re\ s\}, \quad F(s) \text{ analytic in } \Omega$$

needs $e^{-\alpha t} f(t) \to 0, \quad t \to \infty$

- note $\alpha > 0$ allows $f(t) \to \infty, \quad t \to \infty$.

Laplace transform - example

$$f(t) = e^{2t}, \quad F = \mathcal{L}_I\{f\}, \quad F(s) = \lim_{T\to\infty} \int_0^T e^{2t} e^{-st} dt$$

$$F(s) = \lim_{T\to\infty} \left[\frac{1}{2-s} e^{(2-s)t}\right]_0^T = \frac{1}{2-s} \lim_{T\to\infty} \left\{e^{(2-s)T} - 1\right\}$$

$$\lim_{T\to\infty} e^{(2-s)T} = 0, \quad Re\, s > 2$$

So

$$F(s) = \frac{1}{s-2}, \quad Re\, s > 2$$

Extend domain of definition with analytic continuation to $\mathbf{C} - \{s = 2\}$, only possible such function is $F(s) = \frac{1}{s-2}$

Nice video about analytic continuation:
www.youtube.com/watch?v=sD0NjbwqlYw&t=3s

Transfer functions for causal systems

Weight function

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = [t - \tau \to \tau] = -\int_t^0 h(\tau)u(t-\tau)d\tau$$
$$= \int_0^t h(\tau)u(t-\tau)d\tau$$

$$h(\tau), \quad 0 \le \tau < \infty$$

$$G(s) = (\mathcal{L}_I h)(s)$$

Inverse Laplace transform of convolution gives:

$$Y(s) = G(s)U(s)$$

Laplace transform relations

$$\mathcal{L}_I\left(f'\right) = sF(s) - f(0)$$

Proof: Partial integration gives

$$\begin{aligned}
\mathcal{L}_I\left(\frac{df}{dt}\right) &= \int_0^{\infty} e^{-st}\frac{df}{dt}dt \qquad (*)\\
&= s\int_0^{\infty} e^{-st} f(t)dt + \int_0^{\infty} \frac{d}{dt}(e^{-st}f(t))dt\\
&= s\int_0^{\infty} e^{-st} f(t)dt + \left[e^{-st}f(t)\right]_{t=0}^{\infty}\\
&= sF(s) - f(0)
\end{aligned}$$

(If integrals converge and if $e^{-st}f(t) \to 0$ as $t \to \infty$).

Quiz

What is $\mathcal{L}_I(f'')$?

a $s^2F(s) - f(0)$
b $s^2F(s) - f'(0)$
c $s^2F(s) - sf(0) - f'(0)$
d $s^2F(s) - sf'(0) - f(0)$

Answer

c is correct:

$$\begin{aligned}\mathcal{L}_I(f'') &= sL_I(f') - f'(0) = s(sF(s) - f(0)) - f'(0)\\ &= s^2F(x) - sf(0) - f'(0)\end{aligned}$$

Final Value Theorem - sketch

When $s \to 0$ in $(*)$ we get

$$\int_0^\infty \frac{df}{dt}dt = \lim_{s\to 0} sF(s) - f(0)$$

If the limit value $\lim_{t\to\infty} f(t)$ exists, then this can be written

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

which is the final value theorem

Initial Value Theorem - sketch

If we instead let $s \to \infty$ we have

$$\lim_{s\to\infty}\int_0^\infty e^{-st}\frac{df}{dt}dt = \lim_{s\to\infty} sF(s) - f(0)$$

As $e^{-st} \approx 0$ when $s \to \infty$, this motivates that we should have

$$0 = \lim_{s\to\infty} sF(s) - f(0)$$

which is the initial value theorem

Both the final and initial value theorems need conditions to guarantee that the calculations we just did hold.

Initial and Final-value theorems - rational F

Initial Value Theorem Assume the Laplace transform $F(s)$ is rational and strictly proper. Then

$$\lim_{t\to+0} f(t) = \lim_{s\to+\infty} sF(s)$$

Final Value Theorem. Assume that $F(s)$ is rational and all poles to $sF(s)$ have negativ real part, then

$$\lim_{t\to+\infty} f(t) = \lim_{s\to+0} sF(s)$$

Transients and initial conditions

$$\begin{aligned}\dot{x} &= Ax + Bu, \qquad x(0) = x_0\\ y &= Cx + Du\end{aligned}$$

Laplace transform gives

$$\begin{aligned}sX(s) - x_0 &= AX(s) + BU(s)\\ X(s) &= (sI - A)^{-1}(BU(s) + x_0)\\ Y &= \underbrace{[C(sI - A)^{-1}B + D]}_{G(s)}U(s) + C(sI - A)^{-1}x_0\end{aligned}$$

Example: Sinusoidal input signal

$$\dot{x} = -x + u \qquad x(0) = x_0 \qquad u(t) = \sin t$$

gives after Laplace transform

$$sX(s) - x(0) = -X(s) + U(s), \qquad U(s) = \frac{1}{s^2+1}$$

Solving for X gives

$$\begin{aligned}X(s) &= \frac{1}{s+1}(U(s) + x_0) = \frac{1}{s+1}\left(\frac{1}{s^2+1} + x_0\right)\\ &= \frac{0.5 - 0.5s}{s^2+1} + \frac{0.5 + x_0}{s+1}\end{aligned}$$

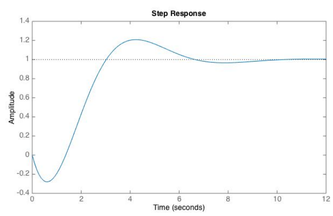
Invers transformation (table) gives

$$x(t) = \frac{1}{2}\sin t - \frac{1}{2}\cos t + (x_0 + \frac{1}{2})e^{-t} = \frac{1}{\sqrt{2}}\sin(t - \frac{\pi}{4}) + (x_0 + \frac{1}{2})e^{-t}$$

sinusoidal in gives sinusoidal out asymptotically

Laplace transform in Matlab (or Maple)

```
>> s=tf('s')
>> G = (1-s)/(s^2+s+1)
G =
      -s + 1
   -----------
   s^2 + s + 1
>> step(G)
```



Laplace transform in Matlab (or Maple)

```
>> clear s
>> syms s t x0

>> ilaplace((1-s)/(s^2+s+1))
ans =
-exp(-t/2)*(cos((3^(1/2)*t)/2) - 3^(1/2)*sin((3^(1/2)*t)/2

>> ilaplace((0.5-0.5*s)/(s^2+1) + (0.5+x0)/(s+1))
ans =
sin(t)/2 - cos(t)/2 + exp(-t)*(x0 + 1/2)

>> latex(ans)
```

$\frac{\sin(t)}{2} - \frac{\cos(t)}{2} + \mathrm{e}^{-t}\left(\mathrm{x0} + \frac{1}{2}\right)$

A sliding block - where will it stop?

A block is sliding according to

$$\ddot{y}(t) + c\dot{y}(t) = 0 \tag{1}$$

with start in position $y(0) = a$ and speed $\dot{y}(0) = b$. Determine $\lim_{t\to\infty} y(t)$.

Laplace transform of (1) gives

$$s^2Y(s) - sy(0) - \dot{y}(0) + c[sY(s) - y(0)] = 0$$

$$Y(s) = \frac{sy(0) + \dot{y}(0) + cy(0)}{s^2 + cs}$$

Final value theorem gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to+0} sY(s) = \lim_{s\to+0} \frac{sy(0) + \dot{y}(0) + cy(0)}{s + c}$$

$$= \frac{\dot{y}(0) + cy(0)}{c} = \frac{b}{c} + a$$

What did we miss? The condition $c > 0$.

Roots and stability

Want to solve the differential equation

$$y^n + a_1 y^{n-1} + \ldots + a_{n-1}y' + a_n y = 0$$

Characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0$$

If $a(\alpha) = 0$ then $y(t) = Ce^{\alpha t}$ is a solution to the differential equation

The general solution is

$$y(t) = \sum_k C_k(t)e^{\alpha_k t}$$

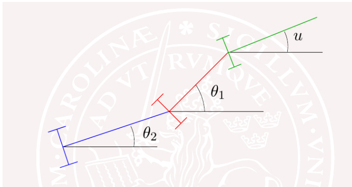
where $C_k(t)$ is a polynomial of degree $m - 1$ if α_k is a root of mult. m

$y(t) \to 0$ if all roots are in the open left half plane

Eigenvalues - stability

$$G(s) = C(sI - A)^{-1}B = \frac{1}{\det(sI - A)} C\mathrm{adj}(sI - A)B$$

Eigenvalues: $\det(sI - A) = 0$.



$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = v\begin{bmatrix}-2 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + v\begin{bmatrix}2\\ 0\end{bmatrix}u$$

How do the eigenvalues depend on speed v?

For what v are the eigenvalues in the open left half plane ?

Frequency analysis

- Frequency curves
  $u(t) = \sin \omega t$, $y(t) = A(\omega)\sin(\omega t + \varphi(\omega))$
  $A(\omega) = |G(i\omega)|$, $\varphi(\omega) = \arg G(i\omega)$
- Representation of $G(s)$ and $G(i\omega)$
- Nyquist diagram - complex number $G(i\omega)$
- Bode diagram – $|G(i\omega)|$ and $\arg G(i\omega)$
  $G = G_1G_2G_3G_4\ldots$

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