Solutions to exam in Systems Engineering and Process Control 2019-06-07

1.

a. The transfer function has the following Laplace transform:

$$sY(s) + 2Y(s) = 3\,U(s)$$

Reordering gives

$$Y(s) = \frac{3}{s+2}U(s)$$

The transfer function is defined through $P(s) = Y(s)/U(s)$ and therefore given by

$$P(s) = \frac{3}{(s+2)}$$

The system has one pole in $s = -2$, but no zeros. Since all poles lie strictly in the left half plane, the system is asymptotically stable. For asymptotically stable systems, the static gain is given by $P(0) = 3$.

b. The P controller has the transfer function $C(s) = K$. The transfer function from reference to process output is give by

$$G(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{\dfrac{3K}{(s+2)}}{1 + \dfrac{3K}{(s+2)}} = \frac{3K}{s + 3K + 2}$$

A first-order system with denominator polynomial $sT + 1$ has time constant T. Division by a static gain does not change the time constant, so it also holds that T is the time constant of a first-order system with denominator polynomial $s + 1/T$. For $T = 1$ we have

$$s + 3K + 2 = s + \frac{1}{0.2} \quad \Rightarrow \quad K = 1$$

The sought P controller is $C(s) = 1$.

c. The time constant of the process is $0.5 = 1/2$, making it slower than the the closed-loop system with time constant 0.2.
Slowing down the closed-loop dynamics results would result in less aggressive responses to disturbances. It could be a good idea if for instance actuator wear or introduction of oscillations are greater concerns than response speed.

2 a. At stationarity, the state variables do not change. That is, their time derivatives are constant:

$$0 = 2\sqrt{x_1} - x_2 + 3\cos(u)$$
$$0 = x_1x_2 - x_1^2 + u$$

Inserting $u = u^0 = 0$ results in

$$0 = 2\sqrt{x_1} - x_2 + 3$$
$$0 = x_1x_2 - x_1^2$$

The second equation has the solutions $x_1 = 0$ and $x_1 = x_2$. Inserting $x_1 = 0$ in the first equation yields $x_2 = 3$. Thus $(x_1^0, x_2^0, u^0) = (0, 3, 0)$ constitutes a stationary point. (Requiring instead that $x_1 = x_2$ in the first equation, yields the other stationary point.)

b. The nonlinear system is on the form $\dot{x} = f(x,u)$, where

$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix}$$

When linearizing, we consider deviations from the stationary point and therefore introduce new variables, which express these deviations:

$$\Delta x = x - x^0$$
$$\Delta u = u - u^0$$

For the dynamics of the new states it holds that $\Delta\dot{x} = \dot{x} - \dot{x}^0 = \dot{x}$. We next approximate the dynamics $\dot{x}$ by the linear portion of their Taylor series expansion at the stationary point $x^0 = (x_1^0, x_2^0)$. This is the linearization step:

$$\Delta\dot{x} = \dot{x} = f(x,u) \approx f(x^0,u^0) + \frac{\partial}{\partial x} f(x^0,u^0) + \frac{\partial}{\partial u} f(x^0,u^0) = A\Delta x + B\Delta u$$

We have that

$$\frac{\partial}{\partial x} f(x,u) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{x_1}} & -1 \\ -2x_1 + x_2 & x_1 \end{bmatrix}$$

$$\frac{\partial}{\partial u} f(x,u) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} -3\sin(u) \\ 1 \end{bmatrix}$$

Evaluating the above at the stationary point, by inserting $(x_1, x_2, u) = (x_1^0, x_2^0, u^0) = (9, 9, 0)$ yields

$$A = \begin{bmatrix} \frac{1}{3} & -1 \\ -9 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To conclude, the linearized system is given by

$$\Delta\dot{x}_1 = \frac{1}{3}\Delta x_1 - \Delta x_2$$
$$\Delta\dot{x}_2 = -9\Delta x_1 + 9\Delta x_2 + \Delta u$$

c. Stability is determined by the eigenvalues λ of the system matrix

$$A = \begin{bmatrix} \frac{1}{3} & -1 \\ -9 & 9 \end{bmatrix}$$

The eigenvalues solve

$$0 = \det(A - \lambda I) = \begin{vmatrix} \frac{1}{3} - \lambda & -1 \\ -9 & 9 - \lambda \end{vmatrix} = \left(\frac{1}{3} - \lambda\right)(9 - \lambda) - 9$$
$$= \lambda^2 - \left(9 + \frac{1}{3}\right)\lambda - 6$$

Since the coefficients of λ and λ^0 (the constant term) are negative, we know from the stability conditions in the collection of formulae that the system is unstable.

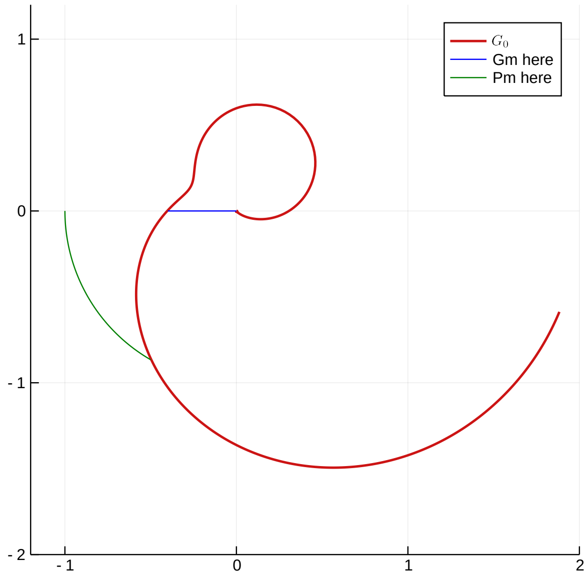

Figure 1 Nyquist plot, solution of Problem 4.

3.
- S2–B1. S2 has unity static gain. The only Bode plot with magnitude 1 for low frequencies is B1.
- S3–B4. S3 does not oscillate, and therefore cannot correspond to B2 or B3.
- S1–B3. S3 has a static gain of 0.1, which matches B3.
- S4–B2. Pigeon hole principle.

4 a. The phase is $\varphi \approx -120^\circ$ at the frequency for which the gain curve crosses 1. The phase margin is therefore $180^\circ + \varphi \approx 60^\circ$.
The gain is $A \approx 0.4$ at the frequency for which the phase curve crosses -180°. The gain margin is therefore $1/A \approx 2.5$.

b. Since the open-loop system is (asymptotically) stable and has positive phase and gain margin, the Nyquist stability criterion dictates that the closed-loop system will become asymptotically stable.

c. See Figure 1. The length of the blue line is $1/A_m$ where A_m is gain margin.
The angle spanned by the green arc, which is part of the unit circle, is the the phase margin.

5 a. We can introduced a name signal Z at the output of the rightmost summation. It holds that

$$\begin{aligned} Z(s) &= U(s) + \alpha Y(s) \\ Y(s) &= \frac{2}{s+2} Z(s) \end{aligned}$$

Substituting the expression for Z into the second equation yields

$$Y(s) = \frac{2}{s+2}\left(U(s) + \alpha Y(s)\right)$$

Solving for Y gives

$$Y(s) = \frac{2}{s+2-2\alpha}U(s)$$

and the sought transfer function of the process is thus

$$P(s) = \frac{Y(s)}{U(s)} = \frac{2}{s+2-2\alpha}$$

b. We know from **a** that the process has stationary gain 2, and that its pole location is governed by the characteristic polynomial $s+2-2\alpha$. For a pole in $s=-1$. this expression must equal $s+1$, which it does for $\alpha = 1/2$.

c. Denoting the input to the controller with E, it holds that

$$U(s) = 5E(s) + \frac{1}{7s+1}U(s)$$

This can be rearranged as

$$5E(s) = \left(1 - \frac{1}{7s+1}\right)U(s) = \frac{7s}{7s+1}U(s)$$

The transfer function fo the controller is

$$C(s) = \frac{U(s)}{E(s)} = 5\frac{7s+1}{7s} = 5\left(1+\frac{1}{7s}\right)$$

This is a PI controller with gain $K=5$ and integral time $T_i = 7$.

d. The closed-loop transfer function from R to Y is given by

$$G(s) = \frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1+P(s)C(s)}$$

This can be assumed known from the course, or computed with the same method used in **a**. Inserting expressions for C and P gives

$$G(s) = \frac{K\left(1+\frac{1}{T_i s}\right)\frac{2}{s}}{1+K\left(1+\frac{1}{T_i s}\right)\frac{2}{s}}$$

This expression simplifies into

$$G(s) = \frac{2K\left(s+\frac{1}{T_i}\right)}{s^2+2Ks+\frac{2K}{T_i}}$$

Since we want both of the poles to lie in $s=-1$ we equate the nominator polynomial with $(s+1)^2 = s^2+2s+1$. The coefficient of s yields $K=1$. This, together with the coefficient of the constant term yields $T_i = 2$.

6. The system is expressed on the standard state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with $B = C = I$, $D = 0$ and A defined in the problem text.

a. The input matrix B has two columns, meaning that u must be a vector with two rows. We thus have two inputs.

b. Since $C = I$ it holds that $Y = X$. Using this and applying the Laplace transform to the fist equation gives $sIY(s) = AY(s) + BU(s)$ and $Y(s) = (sI - A)^{-1}BU(s)$. The sought transfer function matrix is

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D = (sI - A)^{-1} = \begin{bmatrix} s+8 & -1 \\ -1 & s+2 \end{bmatrix}^{-1}$$
$$= \frac{1}{(s+8)(s+2)-1} \begin{bmatrix} s+2 & 1 \\ 1 & s+8 \end{bmatrix} = \begin{bmatrix} \frac{s+2}{s^2+10s+15} & \frac{1}{s^2+10s+15} \\ \frac{1}{s^2+10s+15} & \frac{s+8}{s^2+10s+15} \end{bmatrix}$$

c. The stationary gain matrix is given by

$$G(0) = \frac{1}{15} \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}$$

From $G(0)$ we can calculate

$$\text{RGA} = G(0).*(G(0)^{-1})^T = \frac{1}{15} \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix} .* \left(15 \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}^{-1} \right)^T$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix} .* \left(\frac{1}{15} \begin{bmatrix} 8 & -1 \\ -1 & 2 \end{bmatrix} \right)^T = \frac{1}{15} \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix} .* \begin{bmatrix} 8 & -1 \\ -1 & 2 \end{bmatrix}$$
$$= \frac{1}{15} \begin{bmatrix} 16 & -1 \\ -1 & 16 \end{bmatrix} \approx \begin{bmatrix} 1.07 & -0.07 \\ -0.07 & 1.07 \end{bmatrix}$$

The inputs and outputs should be paired so that the corresponding relative gains are positive and as close to one as possible. Hence, we should control the first output using the first control signal and the second output using the second control signal. The coupling is weak, since the RGA is close to the identity matrix.

7. One possible solution is shown in Figure 2.

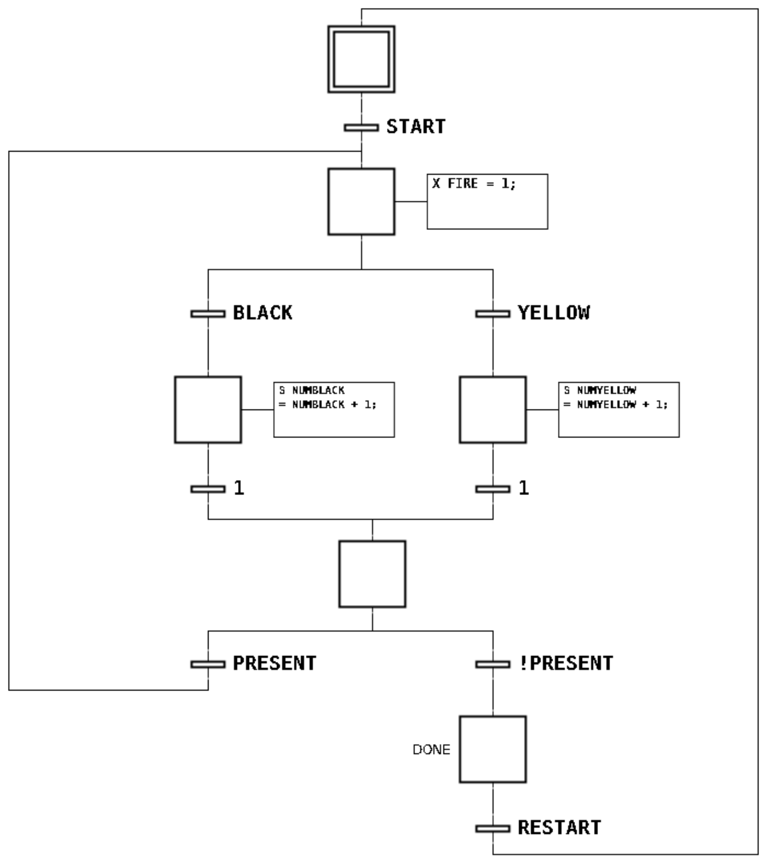

Figure 2 Solution to Problem 7.