Exercises in

Systems Engineering

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1. Basics

1.1 Many situations in everyday life can be described and analyzed using concepts from systems engineering and control theory. Think about the situations described below and try to give a corresponding description that captures the relevant system properties. Which are the measured quantities? Which are the control quantities? Identify possible disturbances, and try to map human behavior to common control structures such as feedback and feedforward. Use block diagrams to capture relevant features of the systems.

a. When you take a shower, you strive to achieve desired temperature and flow. How can this be achieved? What are the relevant measured variables? Control variables?

b. What is the situation when driving a car? What information does the driver use? What can the driver do to affect the behavior of the car? Is feedback or feedforward used? Or both? What conditions can be regarded as disturbances? Draw a block diagram including the car, the driver and relevant signals.

c. When you boil potatoes you are likely to use a stove, a pot and some water to achieve the desired result: tasty boiled potatoes. But what strategy do you use? Think about which measurement and control signals that are involved. What senses do you use?

1.2 a. Figure 1.1 (a) shows a water tank with varying inflow. A person is watching the water level and trying to control it by adjusting the outflow valve. Is this a feedback or feedforward system? What would it be if the person instead is looking at the inflow?

b. In Figure 1.1 (b), the person has been replaced with a sensor, a controller and an actuator. Draw a block diagram that shows how the four elements are interacting.

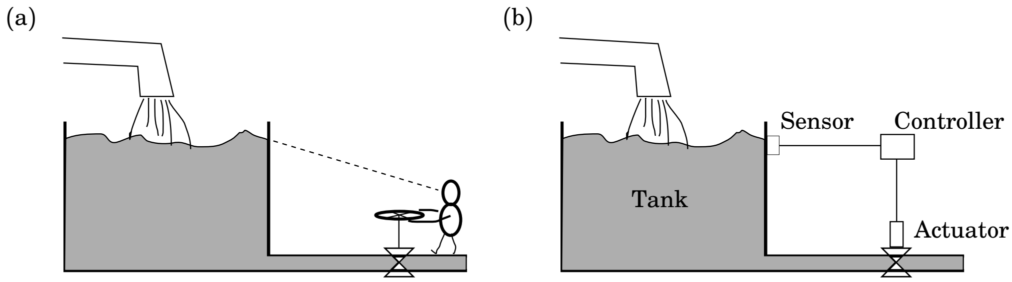


Figure 1.1 (a) Manually controlled water tank. (b) Automatically controlled water tank.

1.3 A tank and its equipment is shown in Figure 1.2. The level in the tank should be controlled. This is done using a controller, a transmitter (= sensor), and an actuator.

Identify the controller, the transmitter, and the actuator in the figure. What is the manipulated process variable, the measured process variable, and the controlled process variable?

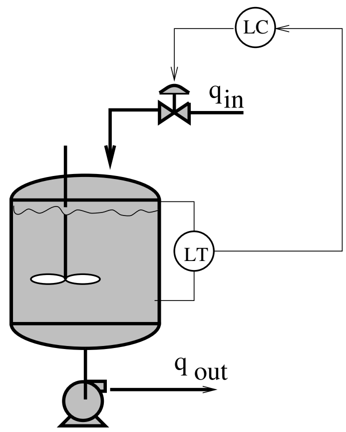


Figure 1.2 The tank system in Problem 1.3.

1.4 Four different processes are shown in Figure 1.3. The measured process variables should in each case be controlled so that the correct value is maintained.

a. What process variable is measured in the different processes? Give some examples of the physical principle on which the transmitters may be based.

b. Explain the relationship that exists between the manipulated process variable and the controlled process variable. Should the valve be opened or closed in order to increase the value of the controlled process variable?

c. Identify possible disturbances that can affect the controlled process variables.

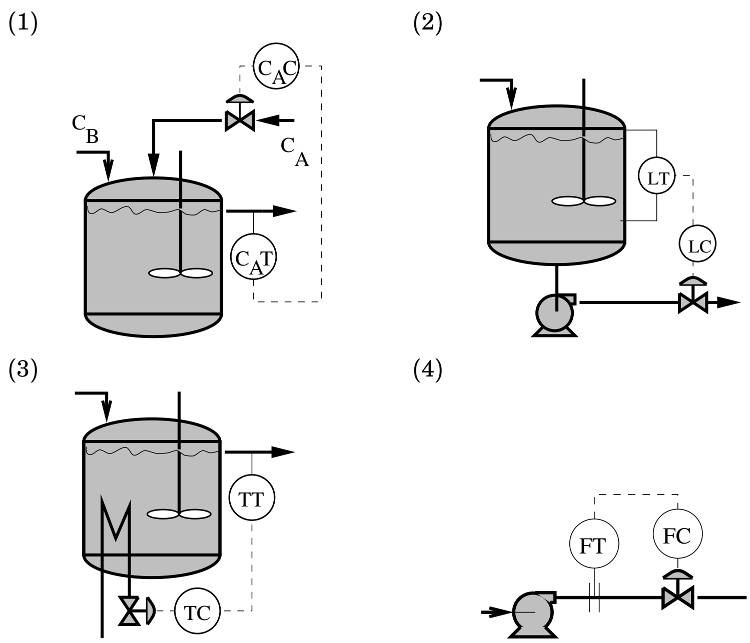


Figure 1.3 Four different processes and their control systems in Problem 1.4. (1) Tank reactor. (2) Buffer tank. (3) Exoterm tank reactor with cooling. (4) Pump with control valve.

1.5 Some new cars have a functionality called adaptive cruise control. Here the normal cruise control has been extended so that if a car with lower velocity is ahead, the speed is adapted to match the other car.

a. Suggest sensors needed for the adaptive cruise control. What is the actuator in this problem?

b. Assume that a sensor that measures the inclination of the road, i.e. uphill or downhill, is available. How can this information be used to increase performance of the cruise control? What is this control strategy called?

1.6 (Extra) Two different control systems are shown in Figure 1.4. Both can be used to control the level in a tank. What control strategy is used in each case? Draw block diagrams for the two control systems.

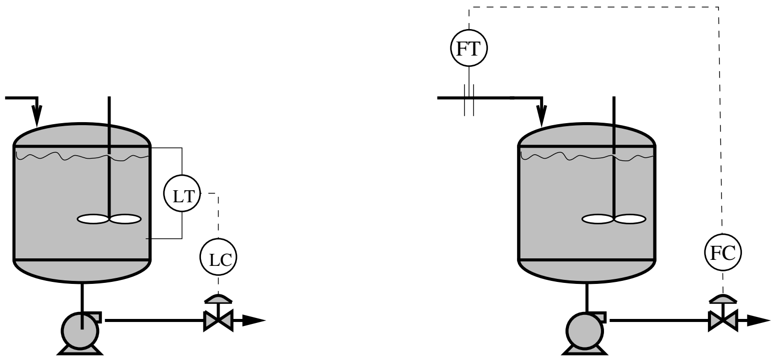


Figure 1.4 Two different control systems used to control a tank in Problem 1.6 (Extra).

1.7 (Extra) A preliminary design of a reactor system is shown in Figure 1.5. The variables that should be controlled are the output flow, level, temperature, and concentration. Variations in the temperature, the concentration of the reactant and the concentration of the solvent are disturbances that may affect the system. Also the temperature of the cooling water may vary.

a. Determine the transmitters needed to measure the process variables.

b. Determine the actuators needed to manipulate the process variables.

c. Suggest how to connect the transmitters, the actuators and the controllers.

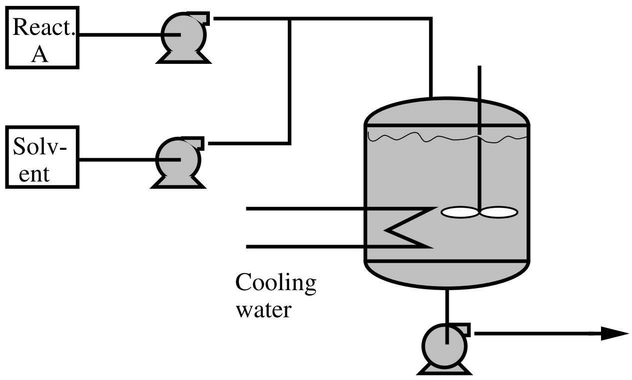


Figure 1.5 A process diagram for the reactor system in problem 1.7 (Extra).

2. Process Models

2.1 For each of the processes in Figure 2.1, derive the process dynamics and write the system in state-space form. What is the order of the system? Is the system linear or nonlinear?

a. A tank with cross section A [m^2], manipulated flows u_1 and u_2 [m^3/s], and measured liquid level y [m]. Let the liquid volume V [m^3] be the state variable of the system.

b. A tank with cross section A [m^2], manipulated inflow u [m^3/s], measured liquid level y_1 [m] and measured outflow y_2 [m^3/s]. By Torricelli's law, the outflow is given by $q_{\text{out}} = a\sqrt{2gh}$, where a [m^2] is the area of the outlet hole, $g = 9.81$ [m/s^2] is the gravitational constant, and h is the height in the tank. Let the liquid volume V [m^3] be the state variable of the system.

c. Two tanks with cross sections A_1 and A_2 [m^2], outlet hole areas a_1 and a_2 [m^2], manipulated inflow u [m^3/s], and measured liquid level y [m]. Let the liquid volumes V_1 and V_2 [m^3] be the state variables of the system.

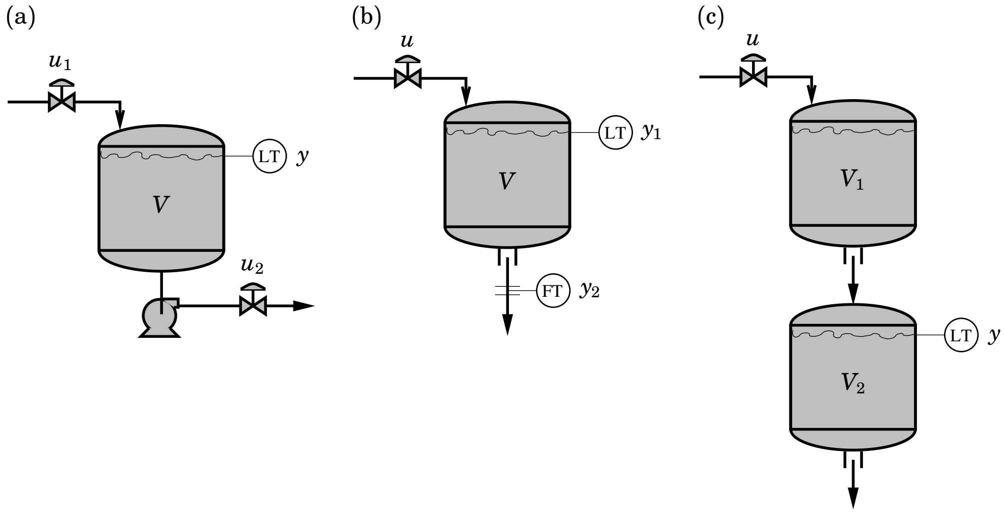


Figure 2.1 The processes in Problem 2.1.

2.2 The moving cart in Figure 2.2 with position x and velocity v is affected by two forces: a spring force $F_s = -kx$ and an external force $F_e = u$.

a. Derive the process dynamics for the cart and write the system in state-space form. Use x and v as state variables, u as the input signal, and $y = x$ as the output signal. Hint: Use Newton's second law of motion: $m\dot{v} = \sum F$.

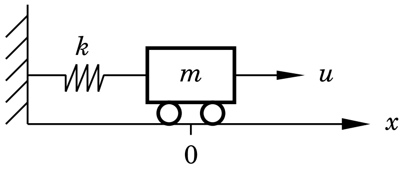


Figure 2.2 The moving cart in Problem 2.2.

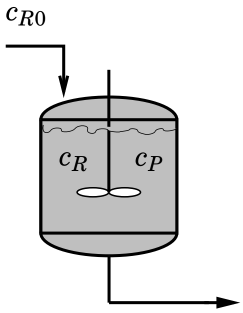


Figure 2.3 Continuous tank reactor in Problem 2.3.

b. Introducing the state vector

$$z = \begin{pmatrix} x \\ v \end{pmatrix},$$

the system dynamics can be written in matrix form as

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz \end{aligned}$$

What are A, B and C?

2.3 Consider a continuous tank reactor with constant volume V, constant flow q, and the two reactants R and P, see Figure 2.3. Derive the process dynamics and write the system in state-space form with matrices. Use the concentrations c_R and c_P as state variables. Let the control signal be the feed concentration c_{R0} and let the measurement signal be c_P.

a. Assume a first-order reaction $R \rightarrow P$, with the reaction rate $r_R = -r_P = -k_1 c_R$.

b. Assume an equilibrium reaction $R \leftrightarrow P$, with the reaction rate $r_R = -r_P = -k_1 c_R + k_2 c_P$.

2.4 Two tanks are connected in series, see Figure 2.4. The tanks have constant volume $V = 1$ and constant flow $q = 1$.

Determine the state-space description for the concentration dynamics in the two tanks. The state variables are the concentrations in the first and second tanks, c_1 and c_2. The input is the feed concentration c_0, and the output is the outflow concentration c_2.

2.5 (Extra) Show that the following physical processes can all be modeled by differential equations of the same form,

$$T \frac{dy(t)}{dt} = -y(t) + u(t),$$

where u is the input, y is the output, and T is the time constant of the system.

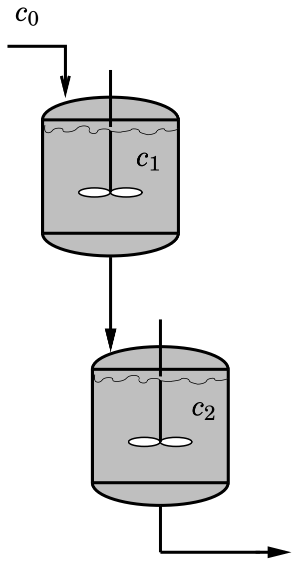


Figure 2.4 The tank system in Problem 2.4.

a. Consider a continuous stirred tank, as seen in Figure 2.5a, where the volume, V, and the flow, q, through the tank are kept constant. Let the feed concentration c_{A0} act as an input, and model the dynamics of the concentration c_A as the output. There is no reaction in the tank.

b. The isothermal process in Figure 2.5b consists of a valve connected to a gas tank with constant volume V. Let the pressure p_0 be the input, and model the tank pressure p_1 as the output. Assume ideal gas behavior and that the mass flow through the valve into the tank is $(p_0 - p_1)/k_v$. Use the gas law $m = Mn = M\frac{pV}{RT}$.

c. A stirred batch reactor is heated by a jacket, see Figure 2.5c. The heat transfer from the jacket to the tank can be described by the static heat transfer equation $Q = kA(T_{jacket} - T_{tank})$. Let the temperature in the jacket, T_{jacket}, be the input, and model the tank temperature T_{tank} as the output. Assume constant mass m and heat capacity $C_p m$ in the tank.

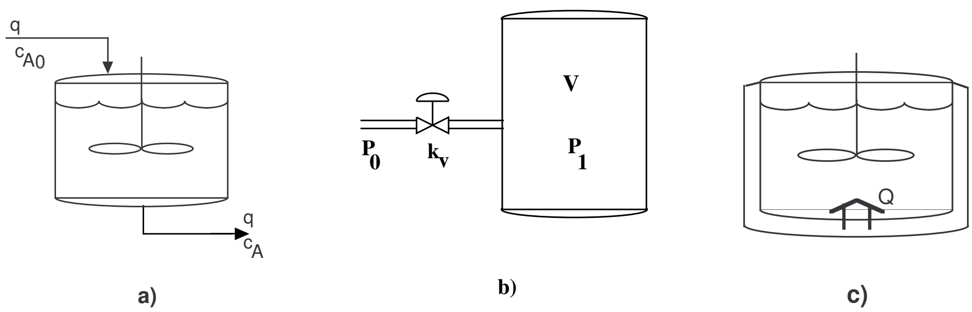


Figure 2.5 The processes in Problem 2.5 (Extra). **a)** A continuous tank. **b)** A gas tank. **c)** A batch reactor with heat transfer.

3. Linear Analysis I

3.1 Figure 3.1 shows a continuous mixing tank. The tank has a constant flow q and a constant volume V. The concentration of a chemical component is described by a first-order differential equation,

$$\frac{V}{q}\,\dot{y} = -y + u,$$

where y is the tank concentration and u is the feed concentration.

a. Calculate the transfer function from u to y.

b. Calculate the poles and zeros of the system. Calculate the impulse response of the system.

c. Sketch the singularity diagram (pole/zero map) and the impulse response when $q = 1$ and $V = 2$.

d. What happens to the singularity diagram and to the impulse response if the flow is increased to $q = 2$, keeping the same volume as before?

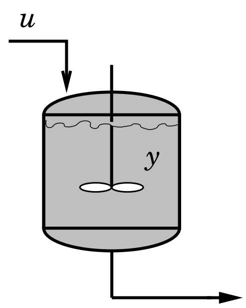


Figure 3.1 The continuous mixing tank in Problem 3.1.

3.2 A linear time-invariant system with input signal $u(t)$ and output signal $y(t)$ is described by the transfer function

$$G(s) = \frac{s+1}{s+2}$$

a. Write down the differential equation that relates the input and output signals.

b. Verify that the system is stable and determine the static gain of the system.

c. Calculate and sketch the step response of the system.

3.3 Calculate the poles (= eigenvalues of the A-matrix) of the following systems. Are the systems stable, asympotically stable, or unstable?

a.

$$\frac{dx}{dt} = -2x + u$$
$$y = 0.5x$$

b.

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} -1 & 1 \end{pmatrix} x$$

c.

$$\frac{dx}{dt} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -3 & 0 \\ 7 & 1 & 2 \end{pmatrix} x + \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} x + 2u$$

3.4 Consider a mercury thermometer measuring the temperature of the surrounding water, see figure 3.2. The temperature of the water is θ_1 and the temperature of the mercury is θ_2. Suppose that the heat capacity of the glass is negligible, i.e. that heat is transferred directly from the water to the mercury. We then get the following differential equation that describes the temperature of the mercury:

$$Mc_p \frac{d\theta_2(t)}{dt} = UA\Big(\theta_1(t) - \theta_2(t)\Big)$$

Here, M is the mass of the mercury, c_p is the heat capacity of mercury, U is the heat transfer coefficient, and A is the heat transfer area.

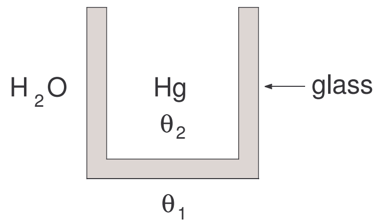


Figure 3.2 The thermometer in Problem 3.4.

a. Find the transfer function from the temperature of the water to the temperature of the mercury.

b. Assume that the system is at rest at time zero: $\theta_1(0) = \theta_2(0) = 0$. Compute and sketch how the temperature of the mercury will change when the temperature of the water changes as a step,

$$\theta_1(t) = \begin{cases} a, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

c. Compute and sketch how the temperature of the mercury will change when the temperature of the water increases linearly,

$$\theta_1(t) = \begin{cases} bt, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

3.5 In Figure 3.3 there are four different processes and four different step responses. Each step response corresponds to one of the processes. Match each process with one of the step responses.

P1 is a tube with ideal plug-flow, where u is the in concentration and y the out concentration.

P2 is a tank with constant outflow, where u is the inflow and y the level in the tank.

P3 is a tank reactor with ideal mixing, where u is the in-concentration of A and y the reactor concentration of B. A first-order reaction $A \longrightarrow B$ occurs in the tank.

P4 is a tank with ideal mixing, where u is the in temperature and y the temperature in the tank.

3.6 (Extra) Consider a process with the transfer function

$$G(s) = \frac{-6s^2 + 6}{s^2 + 5s + 6}$$

a. Calculate the poles and zeros of the process. Is it stable? Calculate the static gain of the process.

b. Assume that the process is at rest at time zero. Calculate the output signal when the input is

$$u(t) = e^t, \qquad t \geq 0$$

In particular, examine the limit of the output signal when $t \to \infty$. Discuss the result.

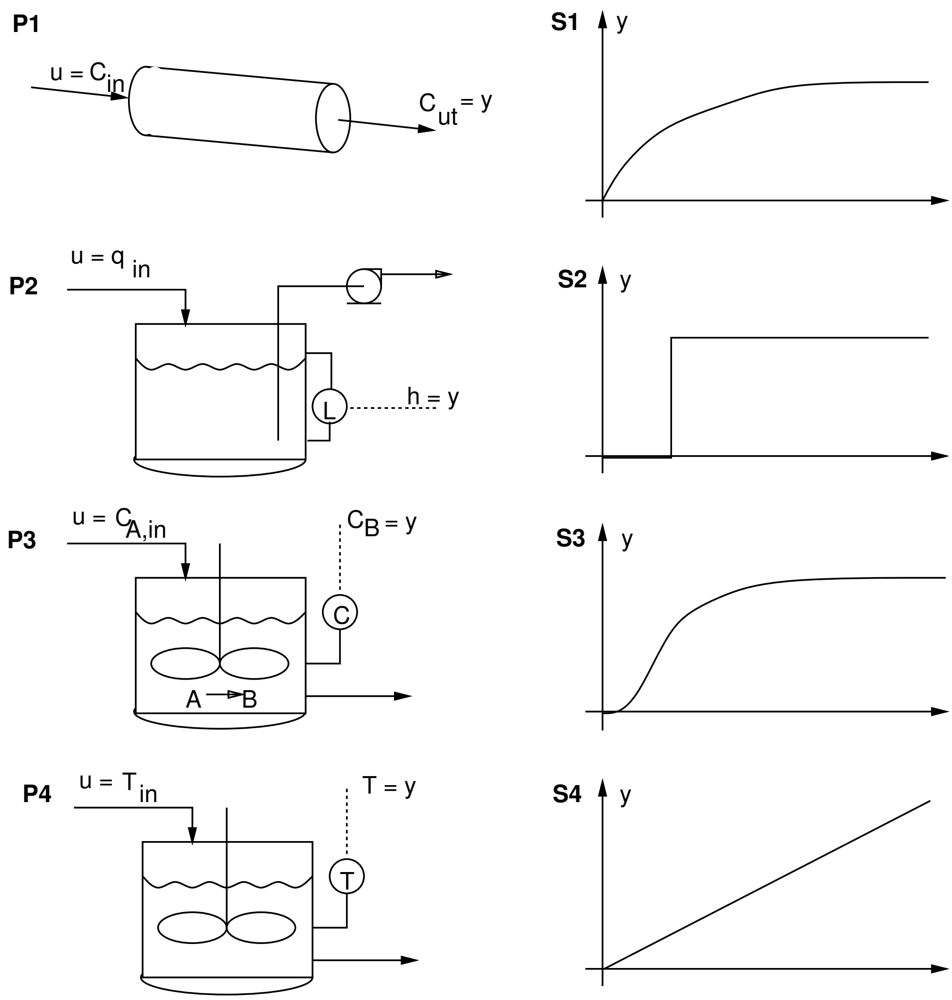


Figure 3.3 Processes and step responses in Problem 3.5

4. Linear Analysis II. MATLAB/Control System Toolbox

4.1 Calculate (by hand) the transfer function, the static gain, the impulse response, and the step response of the following systems:

a.

$$\frac{dx}{dt} = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 5 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} -1 & 1 \end{pmatrix} x + 2u$$

b.

$$\frac{dx}{dt} = \begin{pmatrix} -7 & 2 \\ -15 & 4 \end{pmatrix} x + \begin{pmatrix} 3 \\ 8 \end{pmatrix} u$$
$$y = \begin{pmatrix} -2 & 1 \end{pmatrix} x$$

4.2 Repeat Problem 4.1, but now using MATLAB and Control System Toolbox instead:

- State-space models are created with `ss`.
- Conversion to a transfer function model is done with `tf`.
- The static gain is computed using `dcgain`.
- The impulse response is plotted using `impulse`.
- The step response is plotted using `step`.

Use `help` on the commands above to find out how they are used!

4.3 Consider the three state-space systems in Problem 3.3. Compute the poles and zeros of the systems using MATLAB. Are the systems stable, asymptotically stable, or unstable? Study the behavior of the systems by plotting their impulse responses.

- The poles are computed using `pole`.
- The zeros are computed using `zero`.

4.4 In this problem, we will consider a very simple model of a bicycle. The input to the system is the steer angle δ and the output is the bicycle tilt angle φ. Assuming small angles, a linear model of the system dynamics is given by the transfer function

$$G(s) = \frac{1.3s + 20}{s^2 - 10}$$

a. Enter the bicycle model as a transfer-function model in MATLAB using `tf`. Compute the poles of the system. Is it stable?

b. Plot the step response of the system. Think about the real bicycle—how realistic is the response?

c. In state-space form, the bicycle dynamics can be written as

$$\frac{dx}{dt} = \underbrace{\begin{pmatrix} 0 & 10 \\ 1 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B} \delta$$

$$\varphi = \underbrace{\begin{pmatrix} 1.3 & 20 \end{pmatrix}}_{C} x$$

Enter the bicycle model as a state-space system in MATLAB using `ss`. Plot the pole-zero diagram of the system using `pzmap`.

d. A rider can stabilize the bicycle by applying proportional control,

$$\delta = K(r - \varphi)$$

where K is the feedback gain, and r is the reference value for the tilt angle (typically zero). Show that the resulting closed-loop system becomes

$$\frac{dx}{dt} = (A - BKC)x + BKr$$
$$\varphi = Cx$$

Enter the closed-loop system in MATLAB, assuming $K = 10$. Compute the poles of the system and plot the step response. Does the system response seem to agree with the pole locations?

4.5 (Extra) Figure 4.1 shows the step responses of five different linear time-invariant systems. Each of them corresponds to one of the seven transfer functions below. Pair them up! First reason analytically. Then use MATLAB to verify your conclusions.

I. $G(s) = \dfrac{0.1}{s + 0.1}$

II. $G(s) = \dfrac{4}{s^2 + 2s + 4}$

III. $G(s) = \dfrac{0.5}{s^2 - 0.1s + 2}$

IV. $G(s) = \dfrac{-0.5}{s^2 + 0.1s + 2}$

V. $G(s) = \dfrac{1}{s + 1}$

VI. $G(s) = \dfrac{4}{s^2 + 0.8s + 4}$

VII. $G(s) = \dfrac{2}{s^2 + s + 3}$

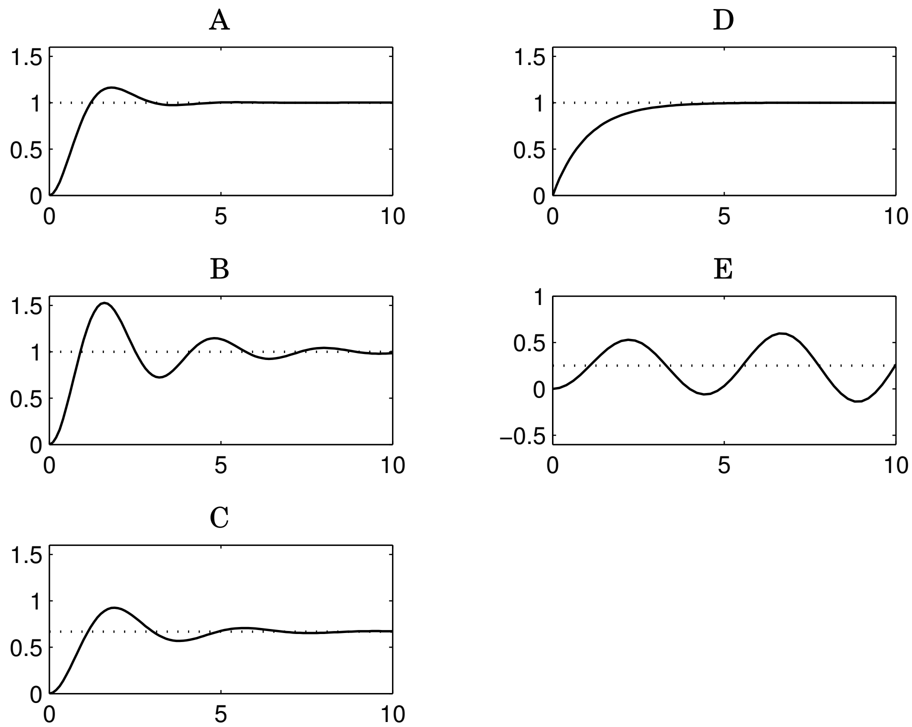


Figure 4.1 Step responses in Problem 4.5 (Extra).

5. Linearization

5.1 Consider the nonlinear dynamical system

$$\dot{x} = -x^3 + u$$
$$y = \sqrt{x}$$

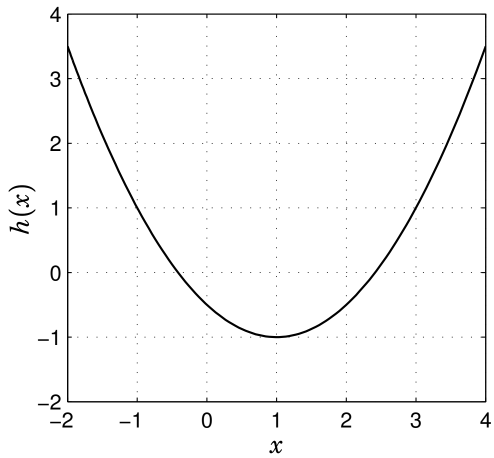
a. Calculate the stationary state x_0 and the stationary output y_0 of the system, given the stationary input $u_0 = 1$.

b. Linearize the system around the stationary point you found in **a**.

5.2 A first-order system is described by the differential equation

$$\dot{x} = h(x) + u$$

where the nonlinear function $h(x)$ is illustrated below:



For each of the cases below,

- find the stationary input u^0 corresponding to the specified stationary state x^0.
- linearize the system around (x^0, u^0).
- determine the stability of the linearized system.

a. $x^0 = -1$

b. $x^0 = 1$

c. $x^0 = 2$

5.3 Consider the Lotka-Volterra system that models the population dynamics of two species

$$\frac{dN}{dt} = N(a - bP)$$
$$\frac{dP}{dt} = P(cN - d)$$

Here, N is the prey population and P is the predator population. The variables a, b, c and d are positive constants.

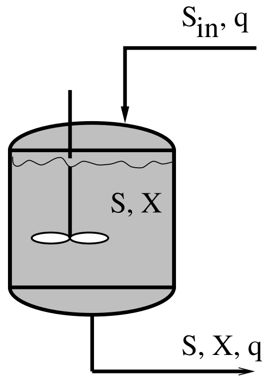


Figure 5.1 The tank reactor in Problem 5.4.

a. Find the two stationary points (N^0, P^0) of the system.

b. Linearize the system around each stationary point.

c. Investigate the stability of each of the linearized systems.

5.4 In a continuous well-stirred biological tank reactor, see Figure 5.1, biomass is growing following Monod kinetics,

$$\mu(S(t)) = \frac{\mu_{\max} S(t)}{K_S + S(t)},$$

where $S(t)$ is the concentration of substrate at time t. The concentration of the incoming substrate is S_{in}. The yield $Y_{X/S}$ describes the amount of substrate that is consumed in relation to the biomass growth.

Assume the following parameter values: $\mu_{\max} = 1\ \text{h}^{-1}$, $Y_{X/S} = 0.5$, $K_S = 0.2$ g/L, $S_{\text{in}} = 10$ g/L and $\frac{q}{V} = 0.5\ \text{h}^{-1}$. The concentration of biomass [g/L] in the tank at time t is denoted $X(t)$.

Biomass and substrate balances give the following nonlinear system:

$$\begin{aligned}
\frac{dX(t)}{dt} &= \left(\mu(S(t)) - \frac{q}{V}\right) X(t) \\
\frac{dS(t)}{dt} &= -\frac{\mu(S(t))}{Y_{X/S}} X(t) + \frac{q}{V}(S_{\text{in}} - S(t))
\end{aligned}$$

a. Determine the stationary points (X^0, S^0) of the system. Compute numerical values of X^0, S^0 and $\mu(S^0)$ in the stationary points.

b. Linearize the system around each stationary point.

c. Determine the eigenvalues of the linearization around each stationary point.

5.5 (Extra) A magnetic levitation device is depicted in Figure 5.2. The purpose of the device is to make the ball float in the air by adapting the voltage

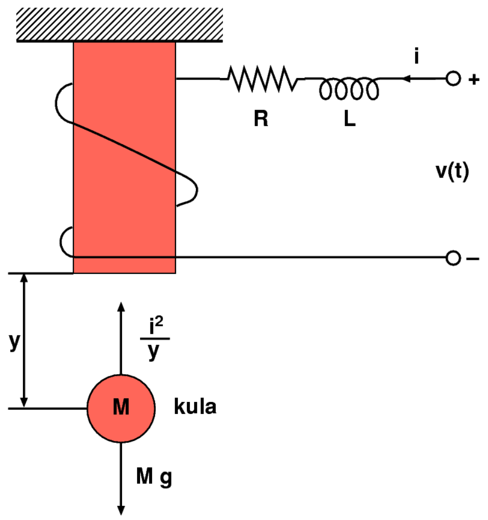


Figure 5.2 Magnetic levitation device

$u(t)$ over the electro-magnet. The system can be described by the nonlinear differential equations

$$M\frac{d^2y(t)}{dt^2} = Mg - \frac{i^2(t)}{y(t)}$$
$$u(t) = Ri(t) + L\frac{di(t)}{dt}$$

where $i(t)$ is the current, R the circuit resistance, L the inductance, $y(t)$ the position of the ball, M the ball mass and g the acceleration due to gravity. In this exercise the mass can be assumed to be $M = 1$.

a. Choose the state variables as $x_1(t) = y(t)$, $x_2(t) = dy(t)/dt$, $x_3(t) = i(t)$, write the equations describing the system dynamics using the new variable names. Let the input signal be the voltage $u(t)$ and let the output signal be the distance $y(t)$.

b. Calculate the equilibrium point for a specified constant distance $y^0 = x_1^0$. What is the required voltage u^0?

c. Linearize the nonlinear equations around the equilibrium point and write the system in matrix state-space form.

6. Simulation using MATLAB/Simulink

Simulink is a MATLAB-based environment for modeling, simulation, and analysis of dynamical systems. It provides a graphical block diagram editor and a large number of block libraries for a variety of different systems. Complete documentation is available on Simulink's homepage,

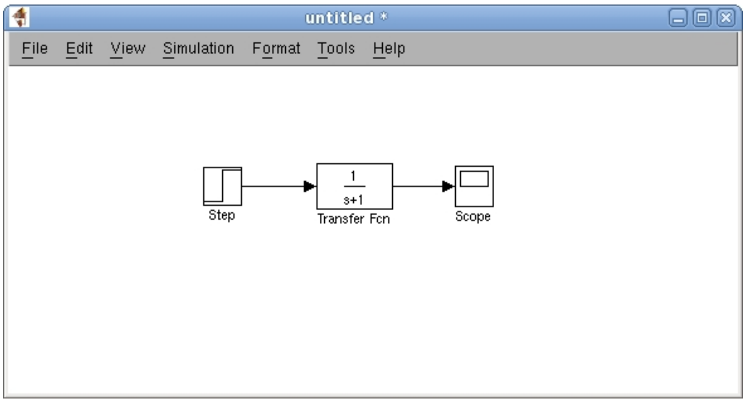
`http://www.mathworks.com/products/simulink/`

6.1 Start MATLAB and type

```
>> simulink
```

The Simulink Library Browser will now appear. Select *File*/*New*/*Model* to create a new model. An empty block diagram will appear.

We will start by modeling a simple linear system. Drag and drop blocks from the Simulink Library Browser into the new model and connect them to create a block diagram like the one below:



The *Step* block is found under *Simulink*/*Sources*, the *Transfer Fcn* block under *Simulink*/*Continuous*, and the *Scope* block is found under *Simulink*/*Sinks*.

Select *Simulation*/*Start* in the menu of the model (or press *Ctrl-t*) to simulate the system for 10 seconds (the default simulation length). Investigate the result by double-clicking on the *Scope* block. Note that you can auto-zoom the plot by clicking on the binoculars.

When does the step in the input occur? What is the final value of the output? Edit the parameters of the *Transfer Fcn* block to instead model the second-order system

$$G(s) = \frac{1}{s^2 + 0.2s + 0.5}$$

and repeat the simulation. Is the result as expected? Select *Simulation*/*Configuration Parameters* and change the stop time to 50 seconds under *Solver*. Repeat the simulation and see that the step response indeed converges to the stationary gain 2.

The resolution of the simulation is a bit too coarse to be satisfactory in this case. Change the relative tolerance to `1e-6` under *Solver* and repeat the simulation. Verify that the output looks smoother now.

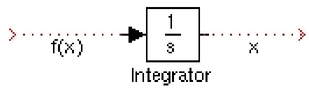
6.2 A great advantage of Simulink compared to Control System Toolbox is that it is possible to model and simulate nonlinear dynamical systems. Note that a nonlinear differential equation

$$\frac{dx(t)}{dt} = f(x(t))$$

can after integration be written as

$$x(t) = x(0) + \int_0^t f(x(\tau))dt$$

In Simulink, $x(t)$ can be found numerically by connecting $f(x)$ to an *Integrator* block like this:



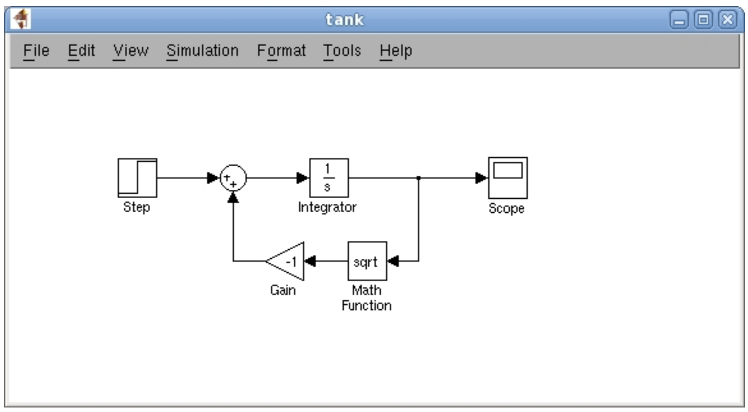
The initial state $x(0)$ can be set by editing the initial condition parameter of the *Integrator* block.

In this problem we will simulate a liquid tank with a free outflow. A normalized model of the process is given by the nonlinear equation

$$\frac{dh}{dt} = -\sqrt{h} + u$$

where h is the height in the tank and u is the inflow.

a. Model the tank system by creating a new Simulink model like the one below:



The *Integrator* block is found under *Simulink / Continuous*. The *Add*, *Gain*, and *Math Function* blocks are available under *Simulink / Math Operations*. Save the model under the name `tank.mdl`.

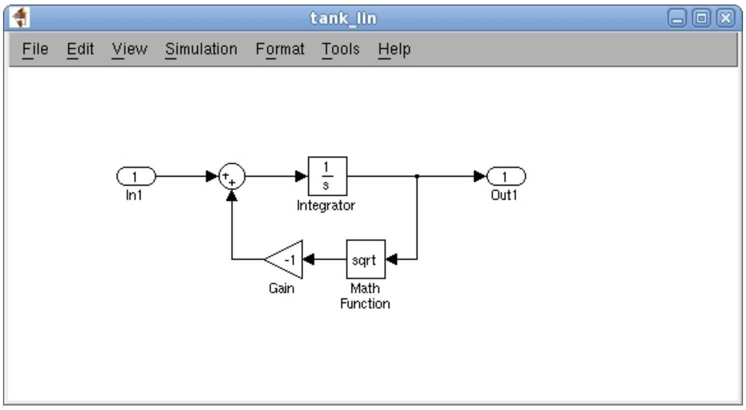
Simulate the system for 10 seconds. What value does h converge to?

b. Run a new simulation where the step change in the input occurs already at time $t = 0$ and where the initial state of the tank is $h(0) = 1$. What is the response now?

c. We now want to linearize the tank system around the stationary point $(h^0, u^0) = (1, 1)$. Assuming a change of variables, $\Delta h = h - h^0$, $\Delta u = u - u^0$, the analytical solution is given by

$$\frac{d\Delta h}{dt} = -\frac{1}{2\sqrt{h^0}}\Delta h + \Delta u$$

In order to use Simulink for linearization, we must modify the model, adding explicit input and output blocks to the model. Change the model so that it looks like below and save it under the new name `tank_lin.mdl`.



The *In1* block is available under *Simulink/Sources*, and the *Out1* block is found under *Simulink/Sinks*.

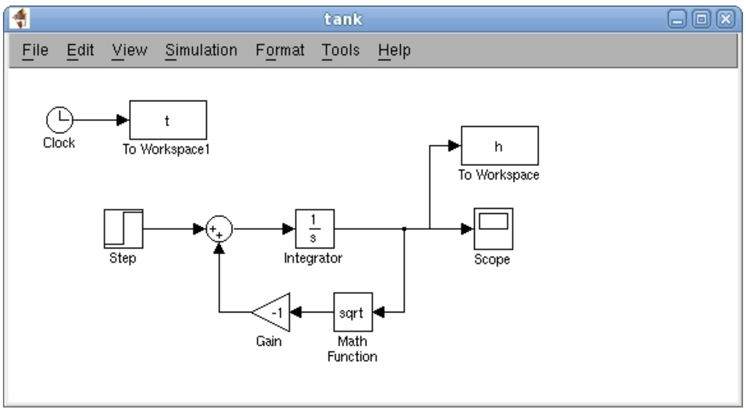
Then use the MATLAB command `linmod` to linearize the model:

```
>> h0 = 1;
>> u0 = 1;
>> [A,B,C,D] = linmod('tank_lin',h0,u0)
```

Does the result agree with the analysis?

d. We now want to compare the step response of the linearized system to the step response of the nonlinear system, drawing both responses in the same MATLAB plot.

Open the original model `tank.mdl` again. In order to save the variables t and h to the MATLAB workspace, we must modify the model like below:



The *Clock* block is found under *Simulink/Sources*, and the *To Workspace* block is found under *Simulink/Sinks*. Edit the parameters of the *To Workspace* blocks, setting the variable names to t and h respectively. Also change the save format to *Array*.

We want the system to start at the stationary point. Hence, set the initial state of the integrator to 1 (the stationary level). Set the *Step time* parameter of the *Step* block to 0, the *Initial value* to 1 (the stationary input) and the *Final value* to 2 (the stationary input plus one).

Simulate the system for 20 seconds and then plot the response:

```
>> plot(t,h)
```

Finally, compute and plot the step response of the linearized system in the same figure:

```
>> sys = ss(A,B,C,D)
>> [h2,t2] = step(sys,20)
>> hold on
>> plot(t2,h2+h0,'--')
>> legend('Nonlinear model','Linearized model')
```

Note that we must add the stationary level h^0 to the linear step response if we want to compare it to the nonlinear response.

How much does the nonlinear and the linear response differ?

6.3 Consider the continuous biological tank reactor from Problem 5.4. Letting $X(t)$ denote the biomass concentration, $S(t)$ the substrate concentration, and $D(t) = q(t)/V(t)$ the dilution rate, the reactor dynamics is modeled by the nonlinear differential equations

$$\begin{aligned} \frac{dX(t)}{dt} &= \Big(\mu(S(t)) - D(t)\Big)X(t) \\ \frac{dS(t)}{dt} &= D(t)\Big(S_{\text{in}} - S(t)\Big) - \frac{\mu(S(t))}{Y_{X/S}}X(t) \end{aligned}$$

where the growth of biomass is assumed to follow the Monod kinetics

$$\mu(S(t)) = \frac{\mu_{\max}S(t)}{K_S + S(t)}$$

The parameter values for a particular reactor are $\mu_{\max} = 1$, $Y_{X/S} = 0.5$, $K_S = 0.2$, and $S_{\text{in}} = 10$.

a. The nonlinear model has been implemented in the Simulink model `reactor.mdl`, which is available on the home page. The model has one input, $D(t)$, and two outputs/states, $X(t)$ and $S(t)$. Open the model by typing `reactor`. Open up the *Reactor* block and locate the two state variables (integrator blocks). What are their initial values?

b. The model has been configured to start in the initial state

$$\begin{pmatrix} X(0) \\ S(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Simulate the system for 20 seconds. The system appears to converge to a stable equilibrium point (X^0, S^0). Which? Compare with the answer to Problem 5.4 a.

c. Modify the model so that it can be linearized by Simulink, i.e., connect the appropriate input and output ports to the reactor. Save the model in a new name, e.g. `reactor_lin.mdl`.

Then use the command `linmod` to linearize the system around the operating point:

```
>> X0 = ...
>> S0 = ...
>> D0 = ...
>> [A,B,C,D] = linmod('reactor_lin',[X0;S0],D0)
```

What are the poles of the linearized system? Is the linearized system stable? Compare with the answer to Problem 5.4 c.

d. Finally, we want to compare simulations of the linear and nonlinear models. We assume that the initial state is the stable stationary point computed above, and that the input changes from 0.5 to 0.6 at time zero. Simulate the systems for 2 seconds.

Proceed similarly to Problem 6.2 d.

7. Feedback Systems

7.1 Assume that you have three linear systems,

$$G_1(s) = 4, \qquad G_2(s) = \frac{1}{s+3}, \qquad G_3(s) = \frac{1}{s}$$

a. Calculate the transfer function from u to y in Figure 7.1.

b. Calculate the transfer function from r to y in Figure 7.2.

c. Calculate the transfer function from r to e in Figure 7.3.

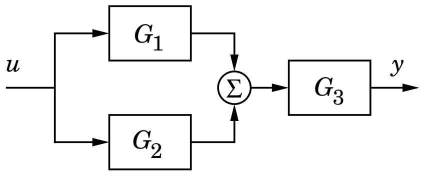


Figure 7.1 A system with parallel and series parts.

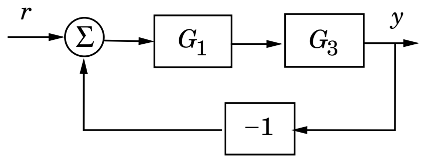


Figure 7.2 A feedback system.

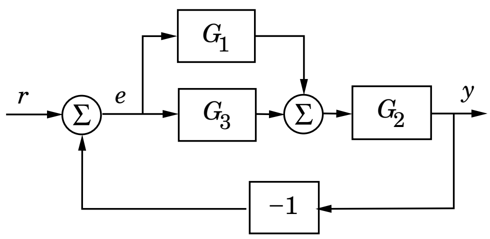


Figure 7.3 A system of parallel, series and feedback parts.

7.2 For what values of k is the closed-loop system in Figure 7.4 stable?

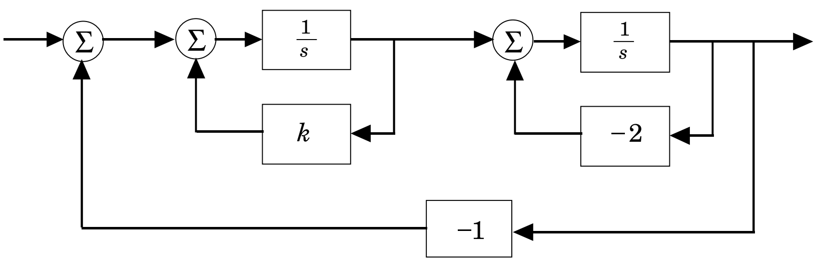


Figure 7.4 Feedback system with stability depending on k.

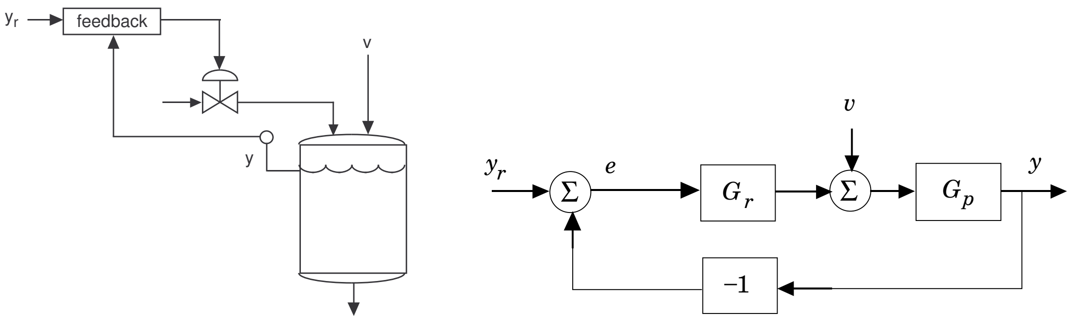


Figure 7.5 Feedback control of a tank process

7.3 A tank process

$$G_p(s) = \frac{b}{s+a}$$

is controlled by a feedback controller $G_r(s)$, see Figure 7.5. The closed-loop system has two inputs: the reference value $y_r(t)$ and the non-measurable load disturbance $v(t)$. Assume that $y_r(t) = v(t) = 1$ and calculate the stationary error $e(\infty)$ in the following cases:

a. $G_r(s) = K$ (P-controller)

b. $G_r(s) = K\left(1 + \frac{1}{sT_i}\right)$ (PI-controller)

7.4 Consider the buffer tank system in Figure 7.6. The inflow q_{in} is determined by a nonlinear valve. The valve position, θ, is the manipulated variable. The inflow q_d is a disturbance. The outflow q_{out} depends on the tank level. The measured process variable is the level in the tank h.

The nonlinear system is linearized around an equilibrium point to give

$$\frac{dy}{dt} = -\frac{1}{T}y + \frac{b}{T}u + \frac{1}{T}v \qquad 0.5 < b < 5$$

where $y = \Delta h$, $u = \Delta\theta$ and $v = \Delta q_d$ are deviations from stationary values. In this problem we want to compare the influence of the disturbance, v, and the varying gain, b, in open loop and in closed loop. We would also like

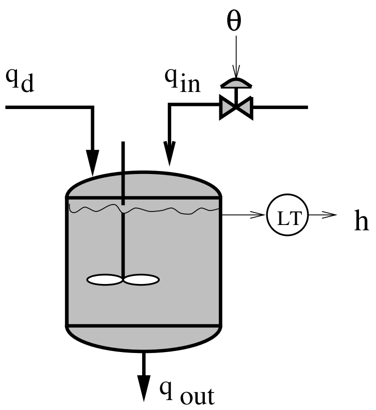


Figure 7.6 A buffer tank

to compare the speed of the system. For the closed-loop case, we assume that we use a P-controller,

$$u = K(y_{\text{ref}} - y)$$

a. Draw block diagrams of the open-loop and closed-loop systems.

b. Calculate the open-loop transfer functions from u and v to y. Also calculate the closed-loop transfer functions from y_{ref} and v to y.

c. Calculate and compare the time constant for the open-loop and the closed-loop systems.

d. Calculate and compare the static gain from v to y in open loop and closed loop, respectively.

e. Calculate the static gain from u to y in open loop and see how the value changes when the parameter b from the nonlinear valve varies from 0.5 to 5. Investigate the ratio between the largest possible static gain and the smallest possible gain. Repeat the same calculations for the closed-loop system, that is the static gain from y_{ref} to y for varying values of b. In the closed-loop case consider also the influence on the static gain due to small and large values of the controller gain K.

7.5 (Extra) The process

$$G_p(s) = \frac{4-s}{(s+2)^3}$$

is controlled with a proportional regulator, $G_r(s) = K$, see Figure 7.7. The process is also affected by a constant load disturbance $v(t) = 1$.

a. Calculate the steady-state error $e(\infty)$ as a function of K assuming $y_r = 0$.

b. Is it possible to choose K such that $|e(\infty)| < 0.1$?

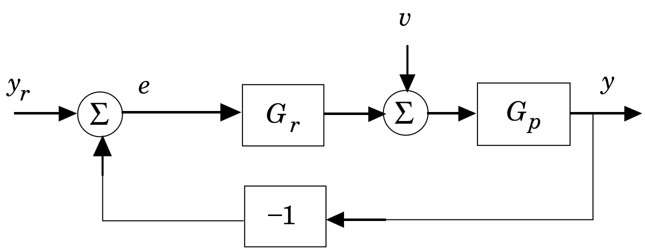


Figure 7.7 Feedback system with load disturbance.

7.6 (Extra) This exercise is supposed to illustrate that successful feedforward control requires a good model of the system. Consider the static system

$$y = G_1 u + G_2 d$$

where u is the input, y is the output, d is a disturbance, and G_1 and G_2 are scalar constants.

a. Design a feedforward controller

$$u = K_1 d + K_2 r$$

that eliminates the influence of d and achieves $y = r$.

b. Assume the nominal parameters $G_1 = 0.1$ and $G_2 = 1$. Determine how the controller designed in **a** works if the real parameters are $G_1 = -0.1$, $G_2 = 1.1$. Comment.

c. Design a feedback controller

$$u = K_1 r + K_2 y$$

that handles the process variations better than the feedforward controller in **a**–**b**.

8. Control Systems Design

8.1 The temperature in a tank reactor is controlled by the temperature in the heating jacket around the reactor vessel. A simple model is a first order transfer function from jacket temperature u to reactor temperature y:

$$Y(s) = \frac{1}{s+1}U(s)$$

A PI-controller

$$U(s) = K\left(1 + \frac{1}{sT_i}\right)E(s)$$

should be designed using pole placement, so that both poles of the closed-loop system are placed in -2.

a. What is the desired closed-loop characteristic polynomial? What natural frequency ω and relative damping ζ does this correspond to?

b. Determine the PI-controller parameters K and T_i so that the specification is fulfilled.

8.2 Solve Preparatory Exercise 3.1 for Laboratory 3, that is, design a PI-controller

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau\right) \quad \Leftrightarrow \quad U(s) = K\left(1 + \frac{1}{sT_i}\right)E(s)$$

for the process

$$G_p(s) = \frac{\rho\tau_1}{1+s\tau_1}$$

so that the closed-loop system gets the characteristic polynomial

$$s^2 + 2\zeta\omega s + \omega^2$$

8.3 Solve Preparatory Exercise 3.8 for Laboratory 3, that is, design a PID-controller

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de(t)}{dt}\right) \Leftrightarrow U(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)E(s)$$

for the process

$$G_p(s) = \frac{\rho\tau_2}{(1+s\tau_1)(1+s\tau_2)}$$

so that the closed-loop system gets the characteristic polynomial

$$(s+\alpha\omega)(s^2 + 2\zeta\omega s + \omega^2)$$

8.4 Use Ziegler–Nichols' step response and frequency method, respectively, to determine the parameters of a PID controller for a system with step response given in Figure 8.1 and Nyquist curve given in Figure 8.2.

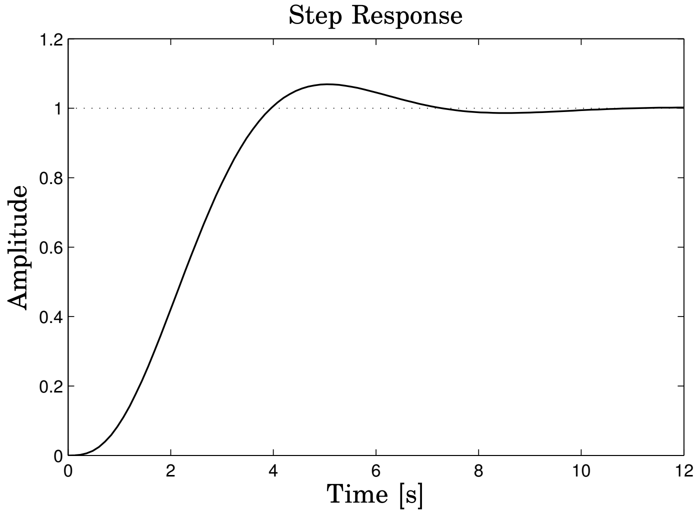


Figure 8.1 Step response of the system in Problem 8.4.

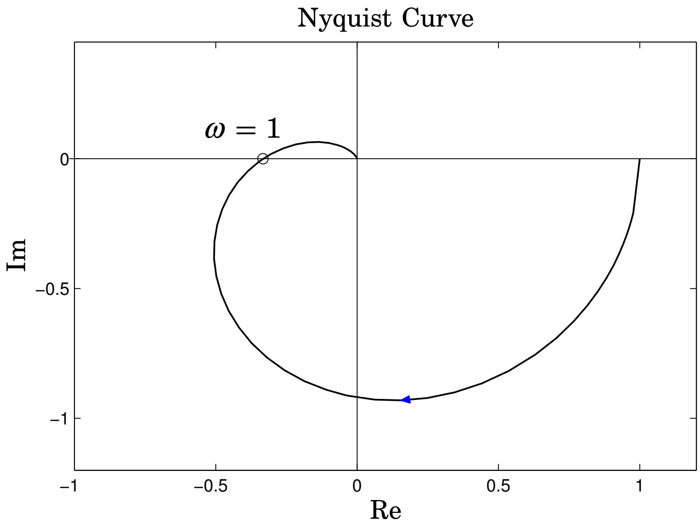


Figure 8.2 Nyquist curve of the system in Problem 8.4.

8.5 (Extra) Consider a system with transfer function

$$G(s) = \frac{1}{(s+1)^3}$$

Calculate the parameters K, T_i and T_d of the PID controller that is obtained when Ziegler–Nichols' frequency method is applied.

9. Analysis in the Frequency Domain

9.1 The relationship between the outdoor temperature and the temperature inside a cave is modeled by the transfer function

$$G(s) = \frac{1}{1 + sT}$$

where the time constant T is 5000 hours.

a. Calculate the frequency function $G(i\omega)$, the amplitude function $|G(i\omega)|$ and the phase function $\arg G(i\omega)$.

b. Assume that the outdoor temperature varies as a sinusoid with the amplitude 5 degrees over one day, and that the day-average temperature varies as a sinusoid with the amplitude 8 degrees over one year. Use the amplitude and phase functions derived in **a** to calculate how the temperature will vary inside the cave.

9.2 Assume that the system

$$G(s) = \frac{0.01(1 + 10s)}{(1 + s)(1 + 0.1s)}$$

is subject to the input $u(t) = \sin 3t, \quad -\infty < t < \infty$

a. Calculate the output $y(t)$.

b. The Bode plot of the system is shown in figure 9.1. Determine the output $y(t)$ (approximately) using the Bode plot instead.

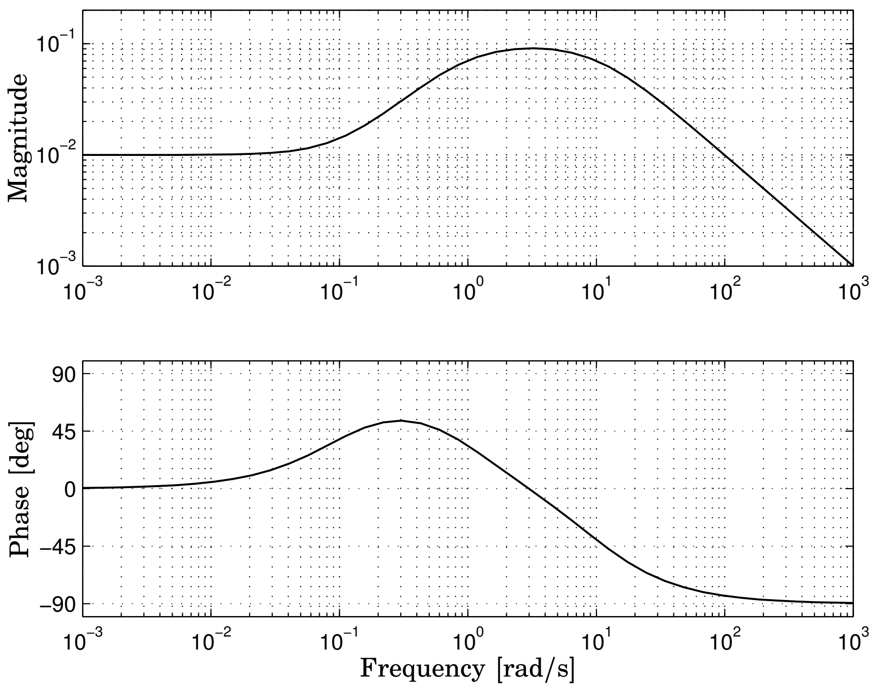


Figure 9.1 The Bode plot in Problem 9.2.

9.3 Consider the Nyquist curves in figure 9.2. Assume that the corresponding systems are controlled by a P-controller

$$u = K(r - y)$$

Which values of K yield stable closed-loop systems? Assume that the open-loop systems are stable.

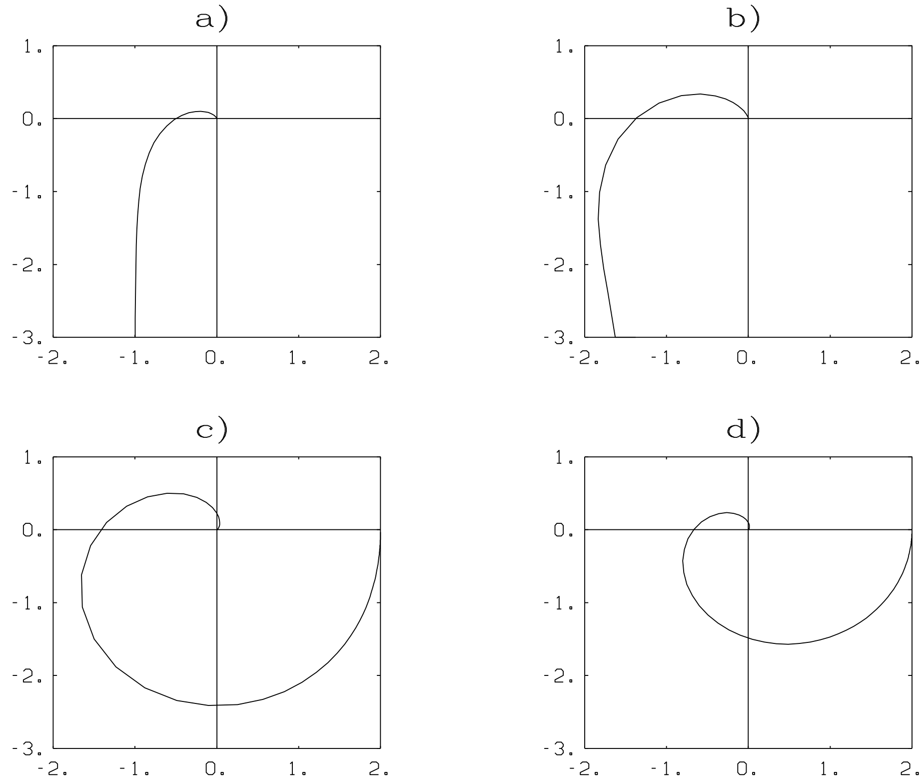


Figure 9.2 Nyquist curves in Problem 9.3.

9.4 The Bode plot and the Nyquist plot of a linear system are shown in Figure 9.3 and Figure 9.4.

a. Determine the amplitude margin of the system using (i) the Bode plot and (ii) the Nyquist plot.

b. Determine the phase margin of the system using (i) the Bode plot and (ii) the Nyquist plot.

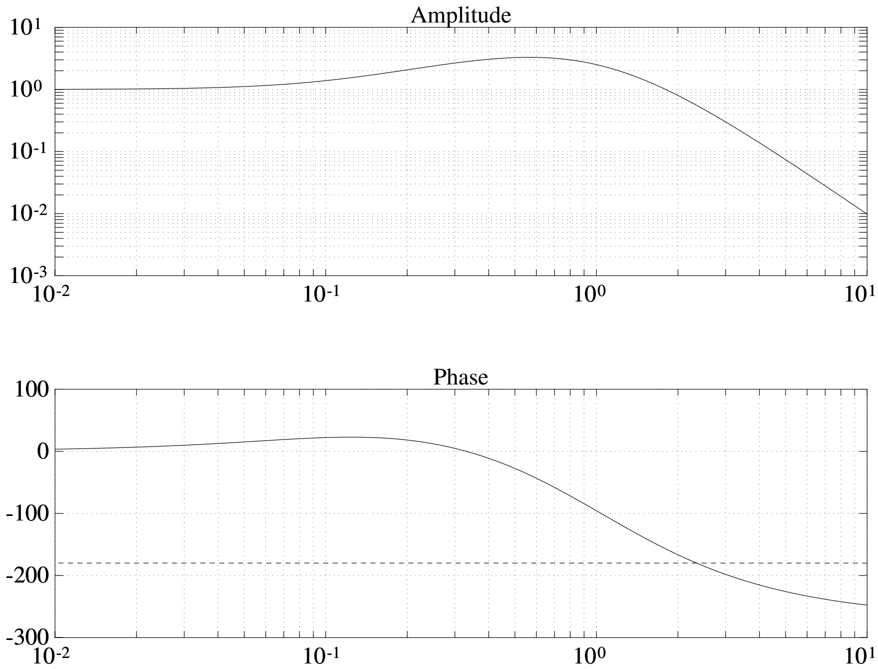


Figure 9.3 Bode plot in Problem 9.4.

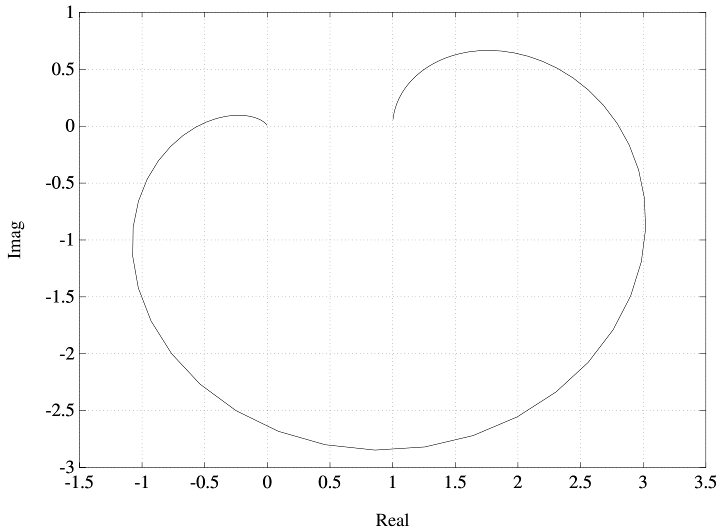


Figure 9.4 Nyquist plot in Problem 9.4.

9.5 A set of frequency response experiments have been performed on a stable system, see Figure 9.5. The dotted curves are the input signal and the solid lines are the output signal of the system. Determine the amplitude and phase margins using the information in the plots.

9.6 (Extra) The Nyquist curve of a system is given in figure 9.6. The system is stable, i.e. lacks poles in the right half plane. Assume that the system is subject to proportional feedback under the control law

$$u = K(r - y)$$

Determine the values of the gain K that yield a stable closed-loop system.

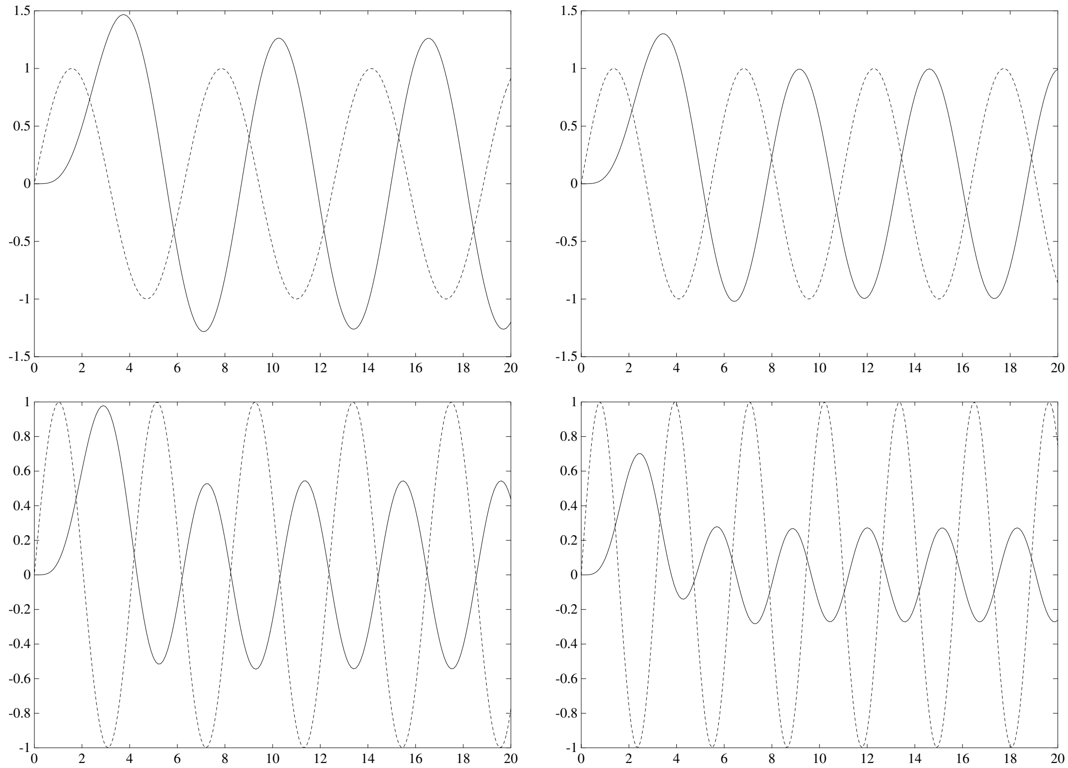


Figure 9.5 The frequency responses in Problem 9.5. The dotted curves are the input u and the solid curves are the system output y.

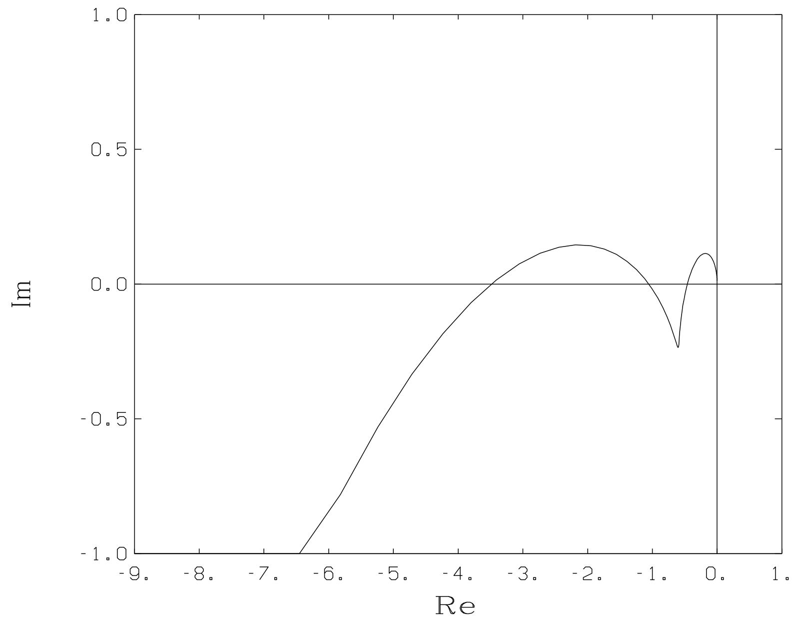


Figure 9.6 Nyquist curve of the system in Problem 9.6 (Extra).

10. Control Structures

10.1 This exercise concerns temperature control of a house. We assume that the temperature in the house, T_h, can be controlled with a heater and that T_h can be measured. T_h is also affected by the outside temperature, T_y, which varies with the time of the day, time of the year etc.

There is a controller connected to the heater that controls the house temperature, see Figure 10.1. The transfer function $G_1(s)$ represents the heater dynamics, $G_2(s)$ describes the temperature dynamics of the house, and $H(s)$ describes how the outer temperature affects the house temperature. $G_r(s)$ can be assumed to be a PI-controller.

The existing PI-control works well but is too slow when fast changes in the outer temperature occur. Therefore, an additional sensor that measures the outer temperature T_y is installed.

Design the feedforward link $G_{ff}(s)$ so that the disturbance T_y does not have an impact on the house temperature. For the calculations the following transfer functions can be assumed:

$$G_1(s) = \frac{1}{s+1} \quad G_2(s) = \frac{1}{s+0.25} \quad H(s) = \frac{1}{s+0.5}$$

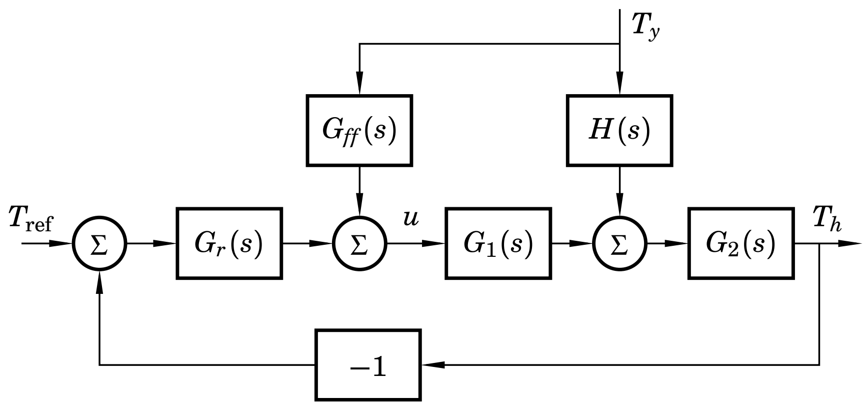


Figure 10.1 House temperature control with feed-forward.

10.2 In this exercise we want to control the level in the reactor shown in Figure 10.2 (top). The reactor level is controlled by the inflow, which is in its turn controlled by the valve. This is a cascade control loop. The outflow disturbance is measured and can therefore be used for feedforward control. The control system is seen in the block diagram in Figure 10.2 (bottom).

a. Design a P-controller for the inner loop that speeds up the valve by a factor five.

b. Design a PI-controller for the outer loop that makes the outer closed loop ten times slower than the inner closed loop. When we have this large difference in time scale between the inner and outer loop, we can approximate the inner loop by its steady-state behaviour: $G_{in}(s) \approx G_{in}(0)$. Also determine the characteristic polynomial for the closed-loop system (without the approximation) and check that it is stable.

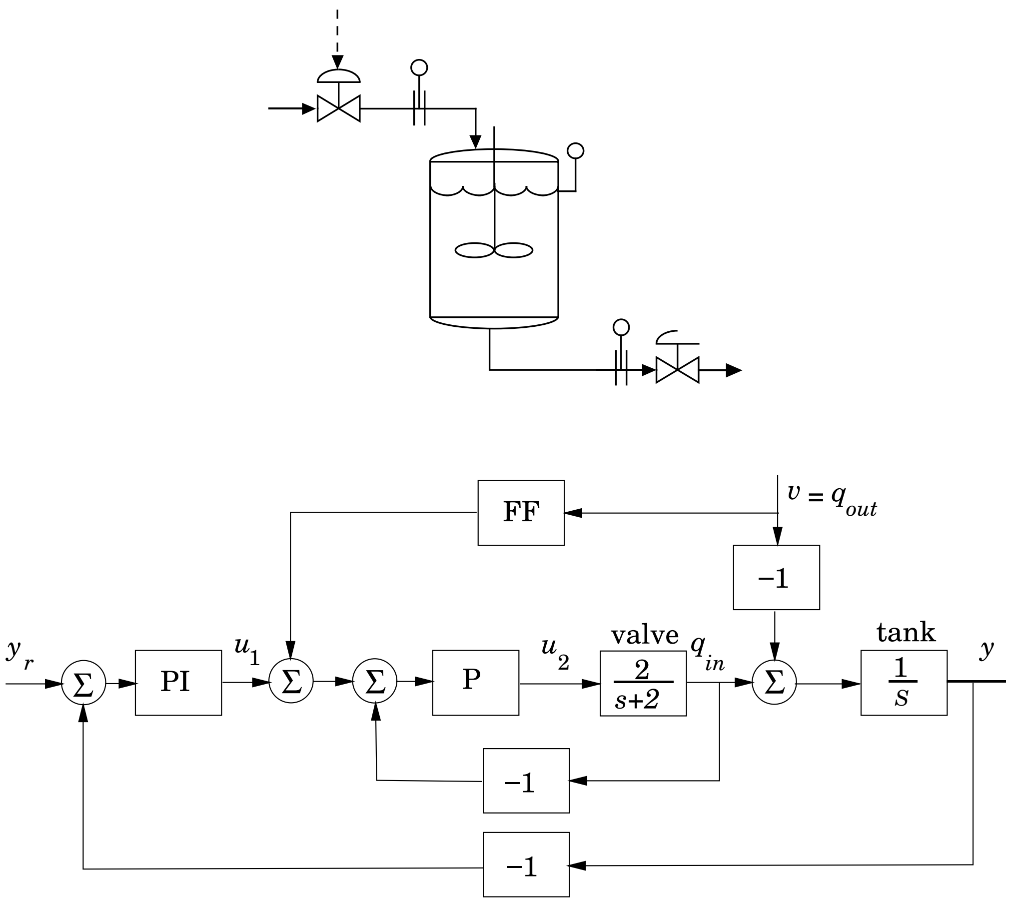


Figure 10.2 The reactor (top) and the control structure (bottom) in Problem 10.2.

c. Design a feedforward controller as indicated in Figure 10.2.

d. Draw the control system in the process flow sheet in Figure 10.2 (top).

10.3 Figure 10.3 shows a process where mid-range control is used to regulate a steam flow. The relationship between the control signals u_1 and u_2 and the steam flow y is given by the relationship

$$Y(s) = \frac{1}{1+s}U_1(s) + \frac{100}{1+10s}U_2(s)$$

a. Draw a block diagram of the system.

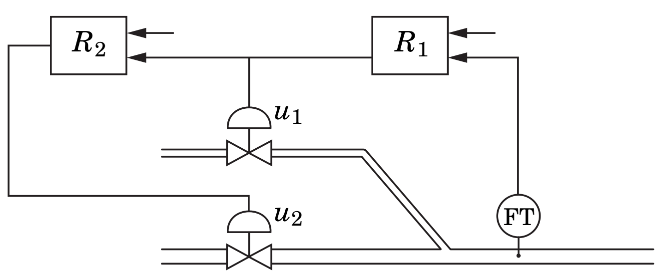


Figure 10.3 Mid-range control of a steam flow.

b. Assume that one controller should be a P-controller and the other should be a pure I-controller. Which one should you choose for R_1 and which for R_2. Explain.

c. Calculate the closed-loop transfer function from y_{ref} (i.e., the reference value of R_1) to y, assuming $K = 1$ and $T_i = 100$ in the controllers. Compute the stationary gain of the closed-loop system.

10.4 (Extra) Figure 10.4 shows a block diagram of a level control system for a tank. The influx $x(t)$ of the tank is determined by the valve position and the outflux $v(t)$ is governed by a pump. The cross section of the tank is $A = 1\ m^2$.

The assignment is to control the system so that the level h in the tank is held approximately constant despite variations in the flow v. The transfer function of the valve from position to flow is

$$G_v(s) = \frac{1}{1 + 0.5s}$$

The tank dynamics can be determined through a simple mass balance.

a. Assume that $G_F = 0$, i.e. that we don't have any feed forward. Dimension a P controller such that the closed loop system obtains the characteristic polynomial $(s + \omega)^2$.

How large does ω become? What stationary level error is obtained after a 0.1 step in $v(t)$?

b. Dimension a PI controller which eliminates the stationary control error otherwise caused by load disturbances.

Determine the controller parameters so that the closed loop system obtains the characteristic polynomial $(s + \omega)^3$. How large does ω become?

c. To further decrease the influence of load disturbances, we introduce a feed forward based on measurements of $v(t)$. Determine a feed forward G_F which eliminates the influence of influx variations by emending $x(t)$.

Note: Since all variables describe deviations from the operation point, the reference value for the level h can be set to zero.

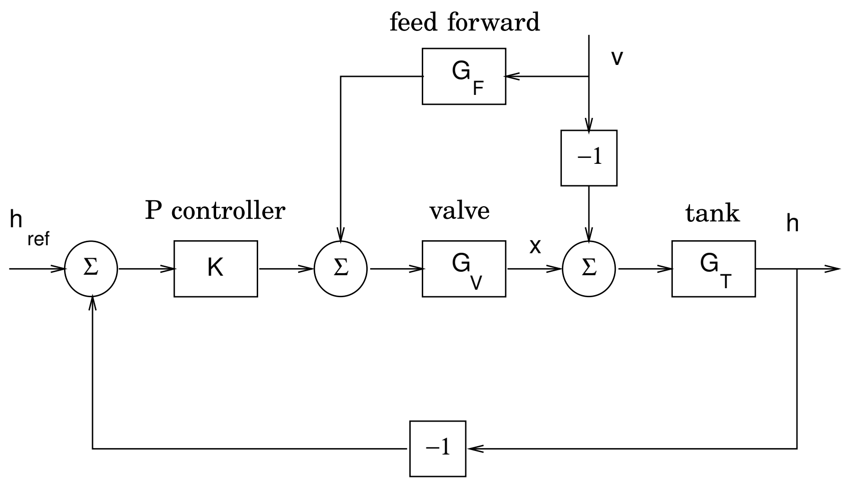


Figure 10.4 Block diagram of the level control system in assignment 10.4 (Extra).

Solutions to Exercise 1. Basics

1.1 a. The situation can be illustrated as in Figure 1.1. The temperature and flow are detected by the skin of the person taking the shower. Using this information, the faucets are used to adjust the water flow and temperature to the desired levels. The situation can be described mainly in terms of feedback.

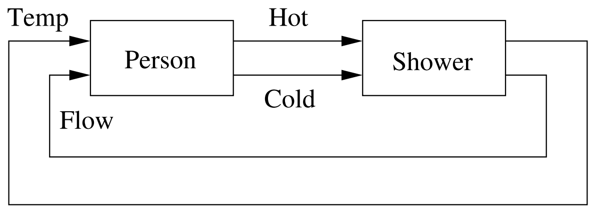


Figure 1.1 Block diagram describing a person in a shower.

b. A driver uses a lot of information when maneuvering a car. The velocity of the car is important, but so is the surroundings of the car. The distance to car in front, curves and road conditions are examples of variables that affect the behavior of the driver. Typical control variables available to the driver are the steering wheel, the accelerator and the brakes. See Figure 1.2.

Feedforward is used extensively by drivers. By observing the surroundings, the driver collects information about what will happen next. For example, if the car is about to enter a curve, the driver is likely to slow down in advance, and then be prepared to turn. Feedback is then used to perform minor adjustments to the position of the car when driving through the curve.

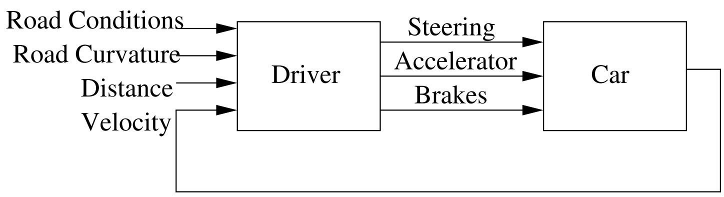


Figure 1.2 Block diagram describing a person driving a car.

c. The scenario can be described as in Figure 1.3. The primary control variable available is the temperature of the hot plate, upon which the pot containing the water and the potatoes is placed. The temperature of the potatoes (which is what we really want to control) is affected only indirectly by the control signal. During the boiling sequence we use different senses to measure relevant quantities. Visual as well as audio information can be used to detect when the boiling starts. We might use a stick or fork to directly measure the state of the potatoes. We are likely to use feedback from different measured variables at different times.

1.2 a. The water level is the variable the person wants to control. By looking, we can measure the water level and affect it with the valve. This is a

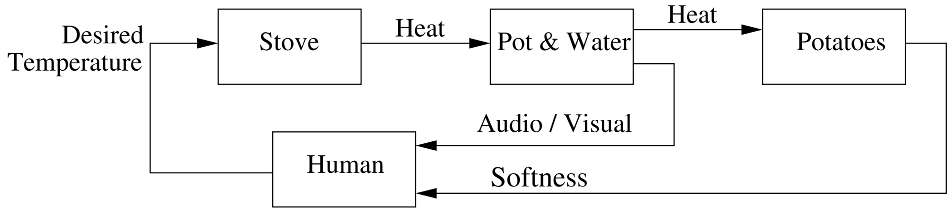


Figure 1.3 Block diagram representation of a person boiling potatoes.

feedback system because we are both measuring and affecting the water level.

If the person instead is watching the inflow, he/she can see the increase before it affects the water level and can faster adapt the valve when an increase of inflow occurs. On the other hand, he/she doesn't know the actual water level. This is a feedforward system, because we are using measurements of a disturbance (the inflow) and trying to compensate for it. We don't affect the inflow with the valve.

b. The block diagram can be drawn as in Figure 1.4. Often, actuators and sensors are not drawn as separate blocks, but are assumed to be included in the process (in this case the tank).

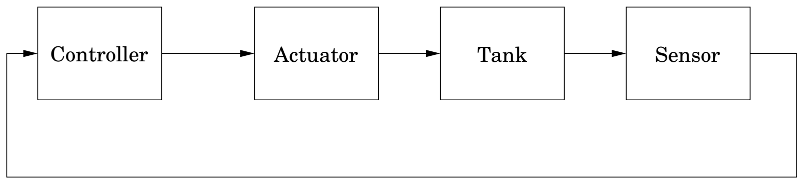


Figure 1.4 Block diagram of water level control.

1.3 Controller = LC (Level Controller)
Transmitter = LT (Level Transmitter)
Actuator = the controlled valve.

Manipulated process variable = the inflow of the tank.
Measured process variable = the level in the tank.
Controlled process variable = the level in the tank.

1.4 1. **Tank reactor:**

(a) The measured process variable is the concentration of substance `A`. The concentration transmitter can be based on measurements of the pH or refractive index.

(b) The manipulated process variable is the inflow of substance `A` and the controlled process variable is the concentration of substance `A`. The inflow to the tank modifies the concentration. By opening the control valve the value of the controlled process variable is increased.

(c) Variations in the concentration of the inflow of substance `B` is a possible disturbance.

2. **Buffer tank:**
   (a) The measured process variable is the level. The level transmitter can be based on measurements of the pressure, measurements of the capacitivity, or it can be given by a float.
   (b) The manipulated process variable is the outflow and the controlled process variable is the level. The outflow affects the level in the buffer tank. By closing the control valve the value of the controlled process variable is increased.
   (c) A possible disturbance is variations in the inflow.
3. **Exothermic tank reactor:**
   (a) The measured process variable is the temperature. The temperature transmitter can be a Hg-thermometer, resistant, semiconductor, thermo-element.
   (b) The manipulated process variable is the heat and the controlled process variable is the temperature. The temperature of the cooling water affects the temperature in the tank. By closing the control valve the value of the controlled process variable is increased.
   (c) A possible disturbance is variations in the temperature of the cooling water.
4. **Pump:**
   (a) The measured process variable is the flow through the valve. The flow transmitter can be based on measurements of the pressure or on an induction transmitter.
   (b) The manipulated process variable is the flow and the controlled process variable is the flow. The inflow affects the outflow. By opening the control valve the value of the controlled process variable is increased.
   (c) Variations in the pump velocity is a possible disturbance.

1.5 a. Possible sensors are

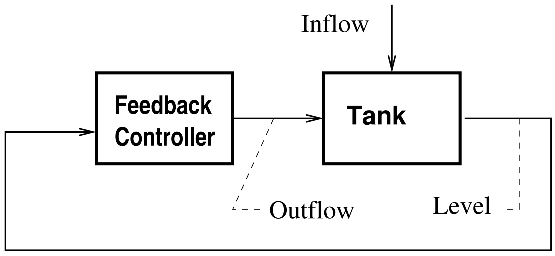
- Car velocity sensor
- Distance sensor, ultrasonic or radar

The actuator is the engine (and possible the brakes).

b. With normal cruise control an incline would be detected by that the velocity decreases, the controller would then increase engine power to correct the speed loss. With the additional information the incline can be detected before any speed loss is detected and therefore higher performance can be achieved. This control strategy is called feedforward.

1.6 (Extra) The left system uses *feedback* from a level sensor to adjust the outflow. The right system uses *feedforward* from a flow sensor in the inflow to adjust the outflow. The two block diagrams are shown in Figure 1.5.

1.7 (Extra) An example of how the reactor system could be controlled is given in Figure 1.6. Several other possibilities exist.



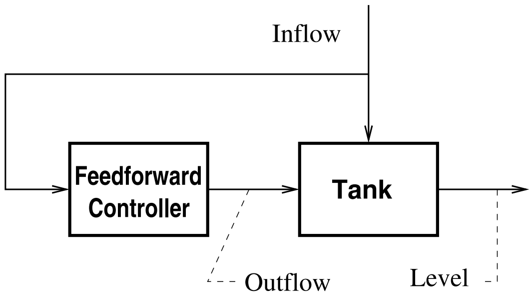


Figure 1.5 Feedback (top) and feedforward (bottom) control of the tank level in Problem 1.6 (Extra).

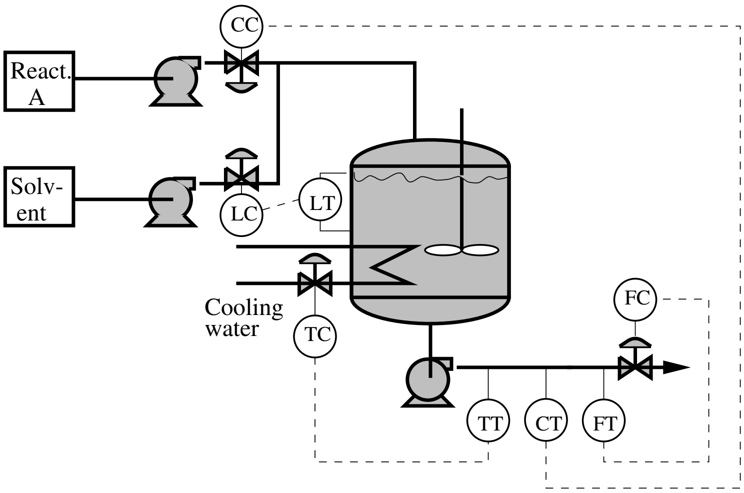


Figure 1.6 A control system for the reactor system in Problem 1.7 (Extra).

a. In order to measure the process variables one transmitter is needed for each variable

- one flow transmitter (FT)
- one level transmitter (LT)
- one temperature transmitter (TT)
- one concentration transmitter (CT)

b. In order to manipulate the process variables, actuators, e.g., valves, are needed.

- A valve placed on the outflow of the reactor can be used to manipulate the outflow of the reactor.

- A valve placed on one of the inflows to the reactor, e.g., the inflow of the solvent, can be used to manipulate the level in the reactor.
- A valve placed on the inflow of cooling water can be used to manipulate the temperature in the tank.
- A valve placed on one of the inflows to the reactor, e.g., the inflow of the reactant, can be used to manipulate the concentration in the reactor.

c. The transmitters, actuators and controllers may for example be connected in the following way

- The flow transmitter (FT) sends its information to a flow controller (FC). The flow controller controls the outflow by opening or closing the valve.
- The level transmitter (LT) is connected to the level controller (LC) which can, by using the valve placed at the solvent, control the level in the reactor.
- The temperature transmitter (TT) sends its information to the temperature controller (TC). The temperature controller can control the temperature in the reactor via the valve.
- The concentration in the reactor is controlled via the valve placed at the reactant A. A concentration controller (CC) controls the valve. The concentration controller gets information about the current concentration from the concentration transmitter (CT).

Solutions to Exercise 2. Process Models

2.1 a. Mass balance in the tank gives the process dynamics

$$
\begin{aligned}
\text{Acc} &= \text{In} - \text{Out} \\
\dot{V} &= u_1 - u_2
\end{aligned}
$$

We can measure $y = \frac{V}{A}$. The complete state-space model is hence

$$
\begin{aligned}
\dot{V} &= u_1 - u_2 \\
y &= \tfrac{1}{A} V
\end{aligned}
$$

This is a linear first-order system.

b. Mass balance in the tank gives the process dynamics

$$
\begin{aligned}
\text{Acc} &= \text{In} - \text{Out} \\
\dot{V} &= u - q_{\text{out}}
\end{aligned}
$$

where $q_{\text{out}} = a\sqrt{2gh} = a\sqrt{\frac{2gV}{A}}$. We can measure $y_1 = \frac{V}{A}$ and $y_2 = q_{\text{out}} = a\sqrt{\frac{2gV}{A}}$. The complete state-space model is hence

$$
\begin{aligned}
\dot{V} &= -a\sqrt{\tfrac{2gV}{A}} + u \\
y_1 &= \tfrac{1}{A} V \\
y_2 &= a\sqrt{\tfrac{2gV}{A}}
\end{aligned}
$$

This is a nonlinear first-order system.

c. Mass balance in the tanks gives the process dynamics

$$
\begin{aligned}
\text{Acc} &= \text{In} - \text{Out} \\
\dot{V}_1 &= u - q_1 \\
\dot{V}_2 &= q_1 - q_2
\end{aligned}
$$

where $q_1 = a_1\sqrt{\frac{2gV_1}{A_1}}$ and $q_2 = a_2\sqrt{\frac{2gV_2}{A_2}}$. We can measure $y = \frac{V_2}{A_2}$. The complete state-space model is hence

$$
\begin{aligned}
\dot{V}_1 &= -a_1\sqrt{\tfrac{2gV_1}{A_1}} + u \\
\dot{V}_2 &= a_1\sqrt{\tfrac{2gV_1}{A_1}} - a_2\sqrt{\tfrac{2gV_2}{A_2}} \\
y &= \tfrac{1}{A_2} V_2
\end{aligned}
$$

This is a nonlinear second-order system.

2.2 a. From Newton's second law of motion, we have that

$$m\dot{v} = -kx + u$$

Further, the derivative of the position is equal to the velocity, i.e.

$$\dot{x} = v$$

The state-space model is hence

$$\begin{aligned}\dot{x} &= v\\ \dot{v} &= -\frac{k}{m}x + \frac{1}{m}u\\ y &= x\end{aligned}$$

b. Introducing the state vector $z = \begin{pmatrix} x \\ v \end{pmatrix}$, the system can be written

$$\begin{aligned}\dot{z} &= \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}}_{A} z + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{B} u\\ y &= \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} z\end{aligned}$$

2.3 Component mass (mole) balance over the tank gives

$$\begin{array}{llll} \text{Acc} & = \text{In} & -\text{Out} & +\text{Prod} \\ V\dot{c}_R & = qc_{R0} & -qc_R & +Vr_R \\ V\dot{c}_P & = & -qc_P & +Vr_P \end{array}$$

a.

$$\underbrace{\frac{d}{dt}\begin{pmatrix} c_R \\ c_P \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} -(\frac{q}{V} + k_1) & 0 \\ k_1 & -\frac{q}{V} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c_R \\ c_P \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} \frac{q}{V} \\ 0 \end{pmatrix}}_{B} \underbrace{c_{R0}}_{u}$$

$$\underbrace{c_P}_{y} = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} c_R \\ c_P \end{pmatrix}}_{x}$$

b.

$$\frac{d}{dt}\begin{pmatrix} c_R \\ c_P \end{pmatrix} = \begin{pmatrix} -(\frac{q}{V} + k_1) & k_2 \\ k_1 & -(\frac{q}{V} + k_2) \end{pmatrix} \begin{pmatrix} c_R \\ c_P \end{pmatrix} + \begin{pmatrix} \frac{q}{V} \\ 0 \end{pmatrix} c_{R0}$$

$$c_P = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} c_R \\ c_P \end{pmatrix}$$

2.4 A component mass balance gives

$$V\frac{dc_1}{dt} = qc_0 - qc_1$$
$$V\frac{dc_2}{dt} = qc_1 - qc_2$$

These equations in matrix form, and with the output-signal $y = c_2$ is in the case with $V = 1$ and $q = 1$ given by

$$\dot{c} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} c(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} c_0(t)$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} c(t)$$

2.5 (Extra)a. Mass (mole) balance over the tank gives

$$\frac{d(Vc_A)}{dt} = qc_{A0} - qc_A$$

This can be rewritten as

$$\frac{V}{q}\dot{c}_A = c_A + c_{A0}$$

b. Mass balance over the gas tank gives

$$\frac{dm}{dt} = q, \text{ where } \; q = (p_0 - p_1)/k_v$$

Rewriting using the gas law results in

$$\frac{MVk_v}{RT}\dot{p}_1 = -p_1 + p_0$$

c. Energy balance over the tank gives

$$\frac{de}{dt} = Q, \text{ where } Q = kA(T_j - T_t)$$

The energy in the tank is described as $e = mC_pT_t$. Rewrite the energy balance

$$\frac{mC_p}{kA}\dot{T}_t = -T_t + T_j$$

Solutions to Exercise 3. Linear Analysis I

3.1 a. Taking the Laplace transform of the differential equation and solving for $Y(s)$ gives

$$\frac{V}{q}sY(s) = -Y(s) + U(s)$$
$$Y(s) = \frac{1}{1 + s\frac{V}{q}}U(s)$$

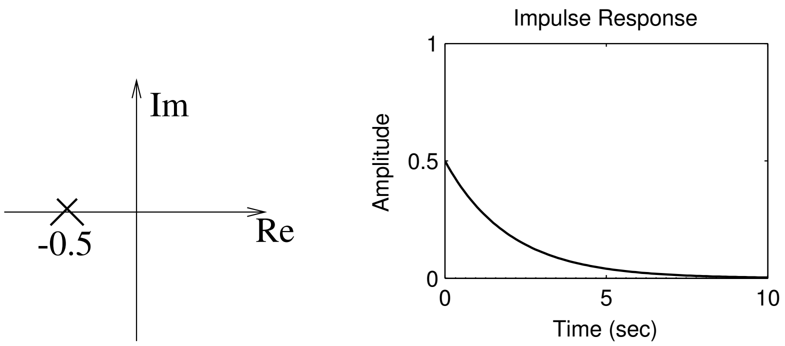
The transfer function from u to y is hence

$$G(s) = \frac{1}{1 + s\frac{V}{q}}$$

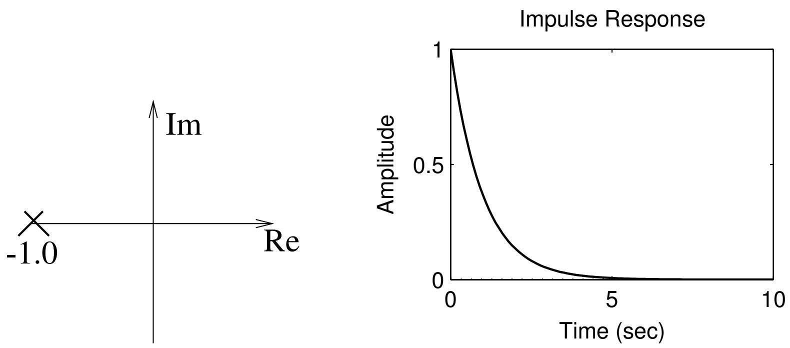
b. The system has no zeros. The system has a single pole in $s = -\frac{q}{V}$.
The impulse response $h(t)$ is given by the inverse Laplace transform of the transfer function:

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{1 + s\frac{V}{q}}\right\} = \frac{q}{V}e^{-\frac{q}{V}t}$$

c. The singularity diagram and the impulse response for $q = 1$ and $V = 2$ are shown below:



d. The singularity diagram and the step response for $q = 2$ and $V = 2$ are shown below:



The distance from the origin to the pole in the singularity diagram is proportional to the decay rate of the impulse response. This means that when the flow increases, the pole moves further away from the origin and the impulse response becomes faster.

3.2 a. In the transfer-function domain, the relation between the input and the output is given by

$$Y(s) = G(s)U(s) = \frac{s+1}{s+2}U(s)$$

We thus have

$$(s+2)Y(s) = (s+1)U(s)$$

Taking the inverse Laplace transform gives

$$\frac{dy}{dt} + 2y = \frac{du}{dt} + u$$

b. The system is stable, since the pole is located in $s = -2$ (i.e., in the left half plane). The static gain is calculated as

$$G(0) = 0.5$$

c. A unit step input has the Laplace transform $U(s) = \frac{1}{s}$.
The output is computed in the Laplace domain as

$$Y(s) = G(s)U(s) = \frac{s+1}{s+2} \cdot \frac{1}{s} = \frac{s+1}{s(s+2)}$$

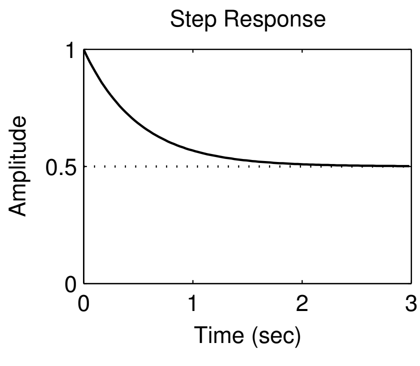
This transform is not available in the collection of formulae. Using a partial fraction decomposition, we can write the output as

$$Y(s) = \frac{1/2}{s} + \frac{1/2}{s+2}$$

Taking the inverse Laplace transform gives

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}$$

The step response is shown below:



It is seen that the step reponse approaches 0.5, which agrees with the static gain of the system.

3.3 a. The eigenvalue of a scalar A-matrix is simply the scalar value itself. The pole of the system is thus -2. Since the pole is located in the left half plane, the system is asympotically stable.

b. The eigenvalues are computed by solving $\det(sI - A) = 0$. We have

$$\det(sI - A) = \det \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix} = s^2 + 1 = 0$$

with the solutions $s = -i$ and $s = i$. Since both poles are located on the imaginary axis, the system is stable (but not asymptotically stable).

c. For a triangular matrix, the eigenvalues are directly identifiable as the diagonal elements. The poles of the system are hence -2, -3 and 2. Since one pole is in the positive half plane, the system is unstable.

3.4 a. To simplify the notation, we introduce the time constant

$$T = \frac{Mc_p}{UA}$$

We have the linear differential equation

$$T\frac{d\theta_2(t)}{dt} = \theta_1(t) - \theta_2(t)$$

Applying the Laplace transform gives

$$Ts\Theta_2(s) = \Theta_1(s) - \Theta_2(s)$$

Solving for $\Theta_2(s)$ gives

$$\Theta_2(s) = \frac{1}{sT+1}\Theta_1(s)$$

The transfer function from θ_1 to θ_2 is hence

$$G(s) = \frac{1}{sT+1}$$

b. The Laplace transform of the input is

$$\Theta_1(s) = \frac{a}{s}$$

The output is computed as

$$\Theta_2(s) = G(s)\Theta_1(s) = \frac{a}{s(1+sT)}$$

Taking the inverse Laplace transform gives

$$\theta_2(t) = a\left(1 - e^{-t/T}\right)$$

A plot of the solution is shown in Figure 3.1. Note that, eventually, the thermometer will show the correct temperature.

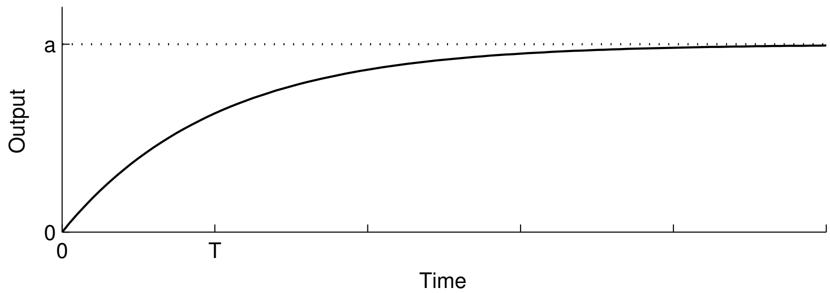


Figure 3.1 The water temperature (dashed line) and the mercury temperature (solid line) as a function of time, when the water temperature is changed as a step at $t = 0$.

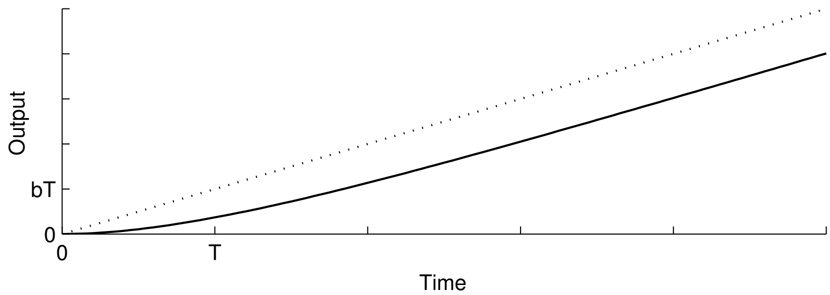


Figure 3.2 The water temperature (dashed line) and the mercury temperature (solid line) as a function of time, when the water temperature is increasing linearly.

c. The Laplace transform of the input is

$$\Theta_1(s) = \frac{b}{s^2}$$

The output is computed as

$$\Theta_2(s) = G(s)\Theta_1(s) = \frac{b}{s^2(1+sT)}$$

Inverse Laplace transform then gives

$$\theta_2(t) = b\left(t - T(1 - e^{-t/T})\right)$$

A plot of the solution is shown in Figure 3.2. Note that there is a stationary error of bT in the temperature measurement.

3.5 **P1** is a pure time-delay. Response **S2**. The same concentration that is flowing in is also flowing out, but at a later time.

P2 is a pure integrator. Response **S4**. The outflow is independent of the level in the tank. If the inflow is greater than the outflow, then the level increases.

P3 is a second order system. Response **S3**. The inflow is mixed into the tank. Then a reaction from A to B occurs. The inflow does not affect the concentration of B directly.

P4 is a first order system. Response **S1**. The inflow is mixed into the tank and it therefore affects the temperature directly.

3.6 (Extra)a. The transfer function can be factored as

$$G(s) = \frac{-6s^2 + 6}{s^2 + 5s + 6} = -\frac{6(s+1)(s-1)}{(s+2)(s+3)}$$

Here we can directly identify the poles and zeros. The poles are located in -2 and -3, i.e., in the left half plane, so the process is stable. The zeros are located in -1 and 1. The static gain is

$$G(0) = 1$$

b. The Laplace transform of the input $u(t) = e^t$ is

$$U(s) = \frac{1}{s-1}$$

The Laplace transform of the output is

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= -\frac{6(s+1)(s-1)}{(s+2)(s+3)} \cdot \frac{1}{s-1} \\ &= \frac{6}{s+2} - \frac{12}{s+3} \end{aligned}$$

Applying the inverse Laplace transform, we have the output

$$y(t) = 6e^{-2t} - 12e^{-3t}$$

When $t \to \infty$, the output approaches zero. We have an exponentially increasing input signal and a positive static gain, but the output signal still goes to zero! This is due to the process zero in the right half plane (also known as an “unstable zero”). In general, a zero in $s = a$ blocks the transmission of the signal e^{at}.

Solutions to Exercise 4. Linear Analysis II. MATLAB/Control System Toolbox

4.1 a. The transfer function is

$$
\begin{aligned}
G(s) &= D + C(sI - A)^{-1}B \\
&= 2 + \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} s+2 & 0 \\ 0 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\
&= 2 + \frac{2}{s+3} - \frac{5}{s+2} \\
&= \frac{2s^2 + 7s + 1}{s^2 + 5s + 6}
\end{aligned}
$$

The static gain is $G(0) = 1/6$.

The impulse response is

$$
h(t) = \mathcal{L}^{-1}\left\{2 + \frac{2}{s+3} - \frac{5}{s+2}\right\} = 2\delta(t) + 2e^{-3t} - 5e^{-2t}
$$

The Laplace transform of the step response is

$$
\begin{aligned}
Y(s) &= G(s)U(s) \\
&= \left(2 + \frac{2}{s+3} - \frac{5}{s+2}\right)\frac{1}{s} \\
&= \frac{2}{s} + \frac{2}{s(s+3)} - \frac{5}{s(s+2)}
\end{aligned}
$$

The step response is

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{2}{s(s+3)} - \frac{5}{s(s+2)}\right\} \\
&= 2 + \frac{2}{3}(1 - e^{-3t}) - \frac{5}{2}(1 - e^{-2t})
\end{aligned}
$$

b. The transfer function is

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B \\
&= \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} s+7 & -2 \\ 15 & s-4 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 8 \end{pmatrix} \\
&= \frac{1}{s+1} + \frac{1}{s+2} \\
&= \frac{2s + 3}{s^2 + 3s + 2}
\end{aligned}
$$

The static gain is $G(0) = 3/2$.

The impulse response is

$$
h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{1}{s+2}\right\} = e^{-t} + e^{-2t}
$$

The Laplace transform of the step response is

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= \left(\frac{1}{s+1} + \frac{1}{s+2} \right) \frac{1}{s} \\ &= \frac{1}{s(s+1)} + \frac{1}{s(s+2)} \end{aligned}$$

The step response is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(\frac{1}{s(s+1)} + \frac{1}{s(s+2)} \right) \\ &= (1 - e^{-t}) + \frac{1}{2}(1 - e^{-2t}) \\ &= \frac{3}{2} - e^{-t} - \frac{1}{2} e^{-2t} \end{aligned}$$

4.2 a.
```
>> A = [-2 0; 0 -3];
>> B = [5; 2];
>> C = [-1 1];
>> D = 2;
>> sys = ss(A,B,C,D)
>> tf(sys)
>> dcgain(sys)
>> impulse(sys)
>> step(sys)
```
Note that the Dirac function at the start of the impulse response is missing in the MATLAB plot!

b.
```
>> sys = ss([-7 2; -15 4],[3; 8],[-2 1],0)
>> tf(sys)
>> dcgain(sys)
>> impulse(sys)
>> step(sys)
```

4.3 a.
```
>> sys = ss(-2,1,0.5,0)
>> pole(sys)
>> zero(sys)
>> impulse(sys)
```
The system is asympotically stable. The impulse response goes to zero.

b.
```
>> sys = ss([0 1; -1 0],[0; 2],[-1 1],0)
>> pole(sys)
>> zero(sys)
>> impulse(sys)
```
The system is stable (but not asymptotically stable). The impulse response is limited but does not go to zero.

c.
```
>> sys = ss([-2 0 0; 1 -3 0; 7 1 2],[5; 2; 0],[0 -1 1],2)
>> pole(sys)
>> zero(sys)
>> impulse(sys)
```
The system is unstable. The impulse response is unlimited.

4.4 a.
```
>> sys = tf([1.3 20],[1 0 -10])
>> pole(sys)
```
The system has one pole in the right half plane and is hence unstable.

b.
```
>> step(sys)
```
The step response shows that the output (the title angle) grows unbounded when the input (the steering angle) is a step. It is realistic that the output deviates, but not that the angle goes towards infinity—a real bicycle will crash when the tilt angle reaches $\pi/2$.

c.
```
>> A = [0 10; 1 0];
>> B = [1; 0];
>> C = [1.3 20];
>> sys = ss(A,B,C,0)
>> pzmap(sys)
```
In the pole-zero map, it is again seen that one pole is in the right half plane.

d. We have the open-loop system

$$\begin{aligned}\frac{dx}{dt} &= Ax + B\delta \\ \varphi &= Cx\end{aligned}$$

The control law is

$$\delta = K(r - \varphi) = Kr - KCx$$

Inserting this into the equations above gives the closed-loop system

$$\begin{aligned}\frac{dx}{dt} &= Ax + B(Kr - KCx) = (A - BKC)x + BKr \\ \varphi &= Cx\end{aligned}$$

e.
```
>> K = 10;
>> syscl = ss(A-B*K*C,K*B,C,0);
>> step(syscl)
>> pole(syscl)
```
The closed-loop system is stable—both poles are located in the left half plane. It can also be noted that the static gain is not completely correct since the output does not converge to 1.

4.5 (Extra) There are several properties to analyze for a dynamical system. First we can look at the stability for each system. All systems except III have poles in the left half plane. System III then corresponds to Figure E, since the amplitude of its oscillations grows. Second, we can look at the steady state values. System VII has a steady state value where $G(0) = \frac{2}{3}$ for $s = 0$, which corresponds to Figure C. There are two systems of order one, I and V, which corresponds to Figure D. By investigating the time constants for the two systems, we see that system V has a time constant of 1, which corresponds to Figure D. The step response for system I would be similar but ten times slower. We can also investigate the slope of the step response and find that the first-order systems have a slope similar to Figure D, whereas all other systems have initial slopes as a parabolic curve. Finally we look at the damping coefficient ζ for the systems II and VI. We then see that system VI is less dampened than system II so system VI corresponds to Figure B. Then system II corresponds to Figure A.

$A \to \mathrm{II}, \quad B \to \mathrm{VI}, \quad C \to \mathrm{VII}, \quad D \to \mathrm{V}, \quad E \to \mathrm{III}.$

Solutions to Exercise 5. Linearization

5.1 We have

$$\begin{aligned} \dot{x} &= -x^3 + u && = f(x,u) \\ y &= \sqrt{x} && = g(x,u) \end{aligned}$$

a. For the stationary point (x_0, u_0) it should hold that $f(x_0, u_0) = 0$. We have

$$\begin{aligned} -x_0^3 + 1 &= 0 \\ x_0 &= 1 \end{aligned}$$

Further, $y_0 = g(x_0, u_0) = 1$.

b. Calculate partial derivatives:

$$\begin{aligned} \frac{\partial f}{\partial x} &= -3x^2 & \frac{\partial f}{\partial u} &= 1 \\ \frac{\partial g}{\partial x} &= \frac{1}{2\sqrt{x}} & \frac{\partial g}{\partial u} &= 0 \end{aligned}$$

Evaluating the derivatives in the stationary point and introducing new variables $\Delta x = x - x_0$, $\Delta u = u - u_0$ and $\Delta y = y - y_0$, we get the linear system

$$\begin{aligned} \dot{\Delta x} &= -3\Delta x + \Delta u \\ \Delta y &= \frac{1}{2}\Delta x \end{aligned}$$

5.2 The general solution is as follows:

- For a stationary point (x^0, u^0) it must hold that

  $$\dot{x} = h(x^0) + u^0 = 0$$

  Hence, for a specified x^0, the corresponding u^0 is given by

  $$u^0 = -h(x^0)$$

- Linearize $h(x)$ by reading the slope $a = \frac{dh}{dx}(x^0)$ from the graph. The equation is already linear in u. After a change of variables,

  $$\Delta x = x - x^0, \quad \Delta u = u - u^0$$

  the linearized system is then given by

  $$\Delta\dot{x} = a\Delta x + \Delta u$$

a. $u^0 = -1$ and $a = \frac{dh}{dx}(x^0) = -2$. The linearized system becomes

$$\Delta\dot{x} = -2\Delta x + \Delta u$$

The system is asymptotically stable, since the eigenvalue (-2) is in the left half plane.

b. $u^0 = 1$ and $a = \frac{dh}{dx}(x^0) = 0$. The linearized system becomes

$$\Delta\dot{x} = \Delta u$$

The system is stable (but not asympotically stable), since the eigenvalue (0) is on the imaginary axis.

c. $u^0 = 0.5$ and $a = \frac{dh}{dx}(x^0) = 1$. The linearized system becomes

$$\Delta\dot{x} = \Delta x + \Delta u$$

The system is unstable, since the eigenvalue (1) is in the right half plane.

5.3 a. In a stationary point (N^0, P^0), it must hold that

$$\begin{aligned} N^0(a - bP^0) &= 0 \\ P^0(cN^0 - d) &= 0 \end{aligned}$$

It is immediately seen that one solution is $(N^0, P^0) = (0, 0)$. If $N^0 \neq 0$ and $P^0 \neq 0$, then

$$\begin{aligned} a - bP^0 &= 0 \\ cN^0 - d &= 0 \end{aligned} \quad \Rightarrow \quad (N^0, P^0) = \left(\frac{d}{c}, \frac{a}{b}\right)$$

b. We should linearize

$$\begin{aligned} \dot{N} &= N(a - bP) && = f_1(N, P) \\ \dot{P} &= P(cN - d) && = f_2(N, P) \end{aligned}$$

After a change of variables, $\Delta N = N - N^0$, $\Delta P = P - P^0$, the linearized system can be written

$$\begin{pmatrix} \Delta\dot{N} \\ \Delta\dot{P} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial N} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial N} & \frac{\partial f_2}{\partial P} \end{pmatrix}_{(N^0, P^0)} \begin{pmatrix} \Delta N \\ \Delta P \end{pmatrix}$$

The partial derivatives are computed as

$$\begin{aligned} \frac{\partial f_1}{\partial N} &= a - bP & \frac{\partial f_1}{\partial P} &= -bN \\ \frac{\partial f_2}{\partial N} &= cP & \frac{\partial f_2}{\partial P} &= cN - d \end{aligned}$$

Around the first stationary point, $(N^0, P^0) = (0, 0)$, we get the linearized system

$$\begin{pmatrix} \Delta\dot{N} \\ \Delta\dot{P} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix} \begin{pmatrix} \Delta N \\ \Delta P \end{pmatrix}$$

Around the second stationary point, $(N^0, P^0) = \left(\frac{d}{c}, \frac{a}{b}\right)$, we get the linearized system

$$\begin{pmatrix} \Delta\dot{N} \\ \Delta\dot{P} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ca}{b} & 0 \end{pmatrix} \begin{pmatrix} \Delta N \\ \Delta P \end{pmatrix}$$

c. For the system linearized around $(0,0)$, we have a diagonal A-matrix where the eigenvalues can be directly identified as a and $-d$. The linearized system is unstable, since there is one eigenvalue, a, in the right half plane.

For the system linearized around $\left(\frac{d}{c}, \frac{a}{b}\right)$, the eigenvalues are given by

$$\det(sI - A) = \det \begin{pmatrix} s & \frac{bd}{c} \\ -\frac{ca}{b} & s \end{pmatrix} = s^2 + ad = 0$$

with the solutions $s = -i\sqrt{ad}$ and $s = +i\sqrt{ad}$. Since the eigenvalues are located on the imaginary axis, the linearized system is stable, but not asympotically stable.

5.4 a. In a stationary point (X^0, S^0) it must hold that

$$\left(\mu(S^0) - \frac{q}{V}\right) X^0 = 0$$
$$\frac{-\mu(S^0)}{Y_{X/S}} X^0 + \frac{q}{V}(S_{\text{in}} - S^0) = 0$$

where

$$\mu(S^0) = \frac{\mu_{\max} S^0}{K_S + S^0}$$

There are two solutions. The first one is

$$X^0 = 0, \quad S^0 = S_{\text{in}} = 10 \text{ g}/\ell, \quad \mu(S^0) = \mu_{\max}\frac{S^0}{K_S + S^0} = 0.98 \text{ h}^{-1}$$

and the second one is

$$\mu(S^0) = q/V = 0.5 \text{ h}^{-1}, \quad S^0 = \frac{\mu(S^0) K_S}{\mu_{\max} - \mu(S^0)} = 0.2 \text{ g}/\ell,$$
$$X^0 = \frac{q Y_{X/S}}{\mu(S^0) V}(S_{\text{in}} - S^0) = 4.9 \text{ g}/\ell$$

b. We should linearize

$$\frac{dX}{dt} = \left(\mu(S) - \frac{q}{V}\right) X \qquad = f_1(X, S)$$
$$\frac{dS}{dt} = -\frac{\mu(S)}{Y_{X/S}} X + \frac{q}{V}(S_{\text{in}} - S) \qquad = f_2(X, S)$$

where

$$\mu(S) = \frac{\mu_{\max} S}{K_S + S}$$

Compute partial derivatives:

$$\frac{\partial f_1}{\partial X} = \mu(S) - \frac{q}{V} \qquad \frac{\partial f_1}{\partial S} = \frac{\partial \mu(S)}{\partial S} \cdot X$$

$$\frac{\partial f_2}{\partial X} = -\frac{\mu(S)}{Y_{X/S}} \qquad \frac{\partial f_2}{\partial S} = -\left(\frac{q}{V} + \frac{X}{Y_{X/S}} \frac{\partial \mu(S)}{\partial S}\right)$$

where

$$\frac{\partial \mu(S)}{\partial S} = \frac{\mu_{\max} \cdot K_S}{(K_S + S)^2}$$

Introducing new state variables $\Delta X = X - X^0$ and $\Delta S = S - S^0$ we finally get the linear system

$$\begin{aligned}
\frac{d\Delta X}{dt} &= \left(\mu(S^0) - \frac{q}{V}\right) \Delta X + \frac{\partial \mu(S^0)}{\partial s} \cdot X^0 \cdot \Delta S \\
\frac{d\Delta S}{dt} &= -\frac{-\mu(S^0)}{Y_{X/S}} \Delta X - \left(\frac{q}{V} + \frac{X^0}{Y_{X/S}} \frac{\partial \mu(S^0)}{\partial s}\right) \cdot \Delta S
\end{aligned}$$

c. For $X^0 = 4.9$, $S^0 = 0.2$ we get

$$\begin{aligned}
\frac{d\Delta X}{dt} &= 6.125 \Delta S \\
\frac{d\Delta S}{dt} &= -\Delta X - 12.75 \Delta S
\end{aligned}$$

with eigenvalues -0.5 and -12.25, i.e., the point is stable.
For $X^0 = 0$, $S^0 = 10$ we get

$$\begin{aligned}
\frac{d\Delta X}{dt} &= 0.48 \Delta X \\
\frac{d\Delta S}{dt} &= -1.96 \Delta X - 0.5 \Delta S
\end{aligned}$$

with eigenvalues -0.5 and 0.48, i.e., the point is unstable.

5.5 (Extra)a. With the new state variables the system can be described by the equations

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= g - \frac{x_3^2(t)}{x_1(t)} \\
\dot{x}_3(t) &= -\frac{R}{L} x_3(t) + \frac{1}{L} u(t) \\
y(t) &= x_1(t)
\end{aligned}$$

b. Given x_1^0 the stationary point in state-space is obtained by setting the derivatives to zero and solving the equations. We get

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix} = \begin{bmatrix} x_1^0 \\ 0 \\ \sqrt{g x_1^0} \end{bmatrix}$$

The required voltage is

$$u^0 = R x_3^0 = R \sqrt{g x_1^0}$$

c. The equations in subproblem **a** can be written as

$$
\begin{aligned}
\dot{x} &= f(x,u) \\
y &= g(x,u)
\end{aligned}
$$

Introduce $\Delta x = x - x^0$, $\Delta u = u - u^0$ and $\Delta y = y - y^0$. The linearized system is then given by

$$
\begin{aligned}
\dot{\Delta x} &= A\Delta x + B\Delta u \\
\Delta y &= C\Delta x + D\Delta u
\end{aligned}
$$

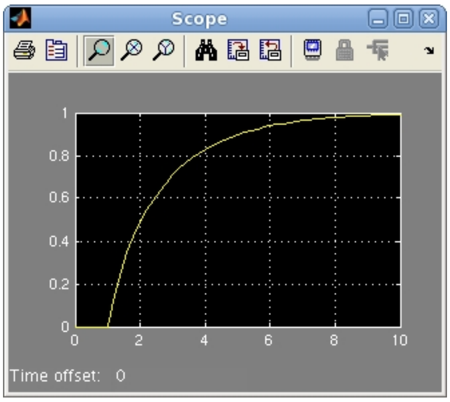
where

$$
\begin{aligned}
A &= \frac{\partial f}{\partial x}(x^0,u^0) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g}{x_1^0} & 0 & -2\sqrt{\frac{g}{x_1^0}} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix} \\
B &= \frac{\partial f}{\partial u}(x^0,u^0) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} \\
C &= \frac{\partial g}{\partial x}(x^0,u^0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
D &= \frac{\partial g}{\partial u}(x^0,u^0) = \begin{bmatrix} 0 \end{bmatrix}
\end{aligned}
$$

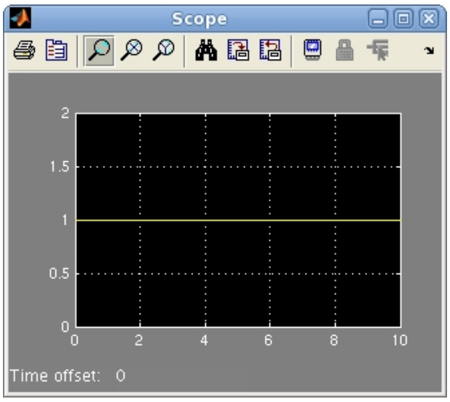
Solutions to Exercise 6. Simulation using MATLAB/Simulink

6.1 –

6.2 a. The tank level converges to 1, see below.

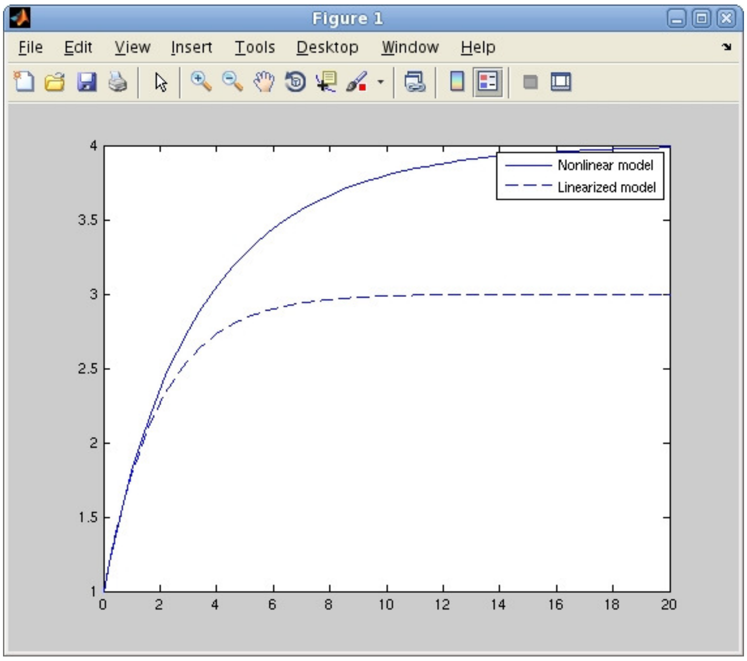


b. The output is constant since the system is in steady state, see below.



c.
```
>> [A,B,C,D] = linmod('tank_lin',h0,u0)
A =
   -0.5000
B =
     1
C =
     1
D =
     0
```
This is exactly what the analysis predicted.

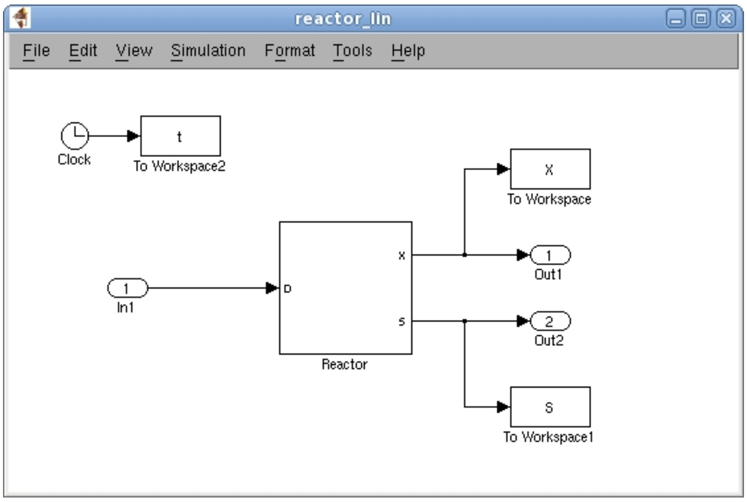
d. The responses have different final values: the nonlinear step response converges to 4, while the linear response converges to 3. The response is very similar in the beginning, though. See the plot below.



6.3 a. –

b. The outputs converge to $X^0 = 4.9$, $S^0 = 0.2$.

c. The modified system is shown below.




```
>> X0 = 4.9;
>> S0 = 0.2;
>> D0 = 0.5;
>> [A,B,C,D] = linmod('reactor_lin',[X0;S0],D0)
A =
         0    6.1250
   -1.0000  -12.7500
B =
   -4.9000
    9.8000
C =
```

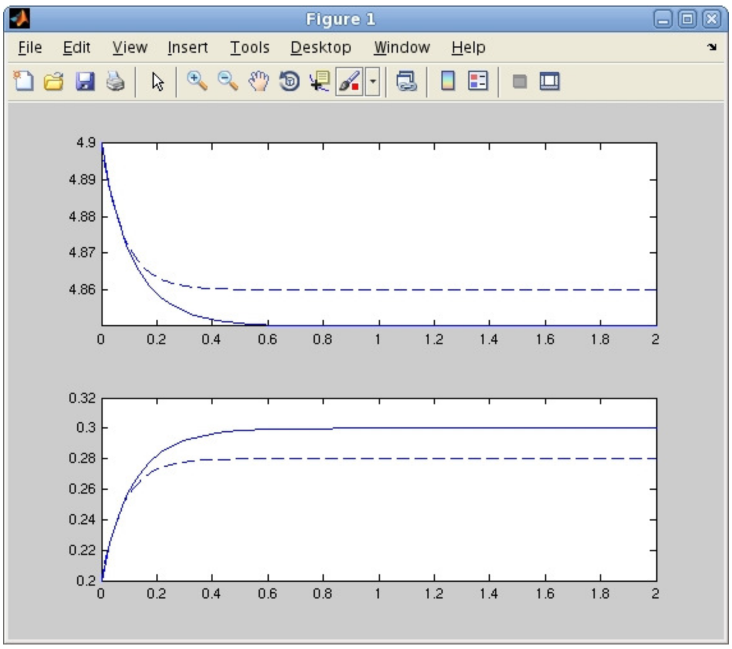
```
     1     0
     0     1
D =
     0
     0
>> eig(A)
ans =
   -0.5000
  -12.2500
```

d. Open the model `reactor.mdl` again. Edit the initial values of the integrators inside the *Reactor* block, setting $X^0 = 4.9$ and $S^0 = 0.2$. Change the value of the constant input to 0.6, or replace the *Constant* block by a *Step* block that makes a change from 0.5 to 0.6 at time 0 (the result will be exactly the same). Set the simulation stop time to 2 seconds and run the simulation.

Create a linear state-space system based on the linearization and compute a step response. Scale the result by 0.1 (the size of the input step), add the stationary values to the outputs, and plot both results in the same figure:

```
>> sys = ss(A,B,C,D);       % create linear model
>> [y2,t2] = step(sys,2);   % compute linear step response
>> X2 = 0.1*y2(:,1) + X0;   % scale X response and add offset
>> S2 = 0.1*y2(:,2) + S0;   % scale S response and add offset
>> subplot(211)
>> plot(t,X);               % plot nonlinear X
>> hold on
>> plot(t2,X2,'--');        % plot linear X
>> subplot(212)
>> plot(t,S);               % plot nonlinear S
>> hold on
>> plot(t2,S2,'--');        % plot linear S
```

The result is shown below.



Solutions to Exercise 7. Feedback Systems

7.1 a.

$$G_{yu}(s) = G_3(s)\Big(G_1(s) + G_2(s)\Big) = \frac{4s+13}{s^2+3s}$$

b.

$$G_{yr}(s) = \frac{G_3(s)G_1(s)}{1+G_3(s)G_1(s)} = \frac{4}{s+4}$$

c.

$$G_{er}(s) = \frac{1}{1+G_2(s)\big(G_1(s)+G_3(s)\big)} = \frac{s^2+3s}{s^2+7s+1}$$

7.2 The closed-loop transfer function is

$$G(s) = \frac{\frac{1}{s-k}\frac{1}{s+2}}{1+\frac{1}{s-k}\frac{1}{s+2}} = \frac{1}{s^2+(2-k)s+(1-2k)}$$

The characteristic equation is

$$s^2+(2-k)s+(1-2k) = 0$$

The system is stable if and only if

$$\begin{cases} 2-k > 0 \\ 1-2k > 0 \end{cases} \quad \Rightarrow \quad k < \frac{1}{2}$$

7.3 The error is given by

$$E(s) = \frac{1}{1+G_r(s)G_p(s)}Y_r(s) - \frac{G_p(s)}{1+G_r(s)G_p(s)}V(s)$$

a.

$$\begin{cases} G_r &= K \\ Y_r(s) &= V(s) = \frac{1}{s} \end{cases} \quad \Rightarrow \quad E(s) = \frac{s+a}{(s+a+bK)}\frac{1}{s} - \frac{b}{(s+a+bK)}\frac{1}{s}$$

The final value theorem gives

$$e(\infty) = \lim_{s\to 0} sE(s) = \frac{a-b}{a+bK}$$

assuming that $a+bK>0$ so that $sE(s)$ is asymptotically stable.

b.

$$\begin{cases} G_r(s) &= K(1+\frac{1}{T_i s}) \\ Y_r(s) &= V(s) = \frac{1}{s} \end{cases} \quad \Rightarrow$$

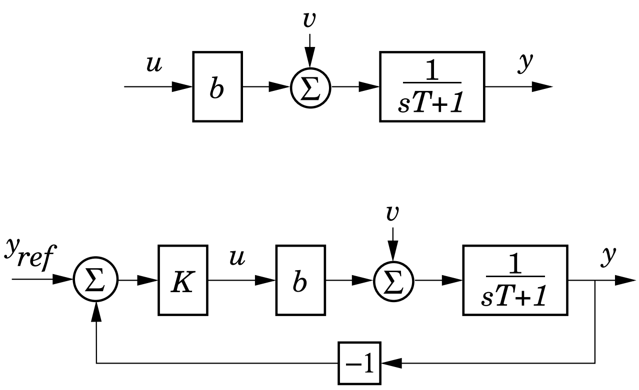


Figure 7.1 The open-loop system (upper) and closed-loop system (lower).

$$E(s) = \frac{s(s+a)}{(s^2+(a+bK)s+bK/T_i)}\frac{1}{s} - \frac{bs}{(s^2+(a+bK)s+bK/T_i)}\frac{1}{s}$$

The final value theorem gives

$$e(\infty) = \lim_{s\to 0} sE(s) = 0$$

if $sE(s)$ is asymptotically stable, i.e., if $a + bK > 0$ and $bK/T_i > 0$.

7.4 a. The open-loop and closed-loop systems are shown in Figure 7.1.

b. In open loop we have

$$Y(s) = \frac{b}{Ts+1}U(s) + \frac{1}{Ts+1}V(s)$$

and in closed loop

$$Y(s) = \frac{Kb}{Ts+1+Kb}Y_{\text{ref}}(s) + \frac{1}{Ts+1+Kb}V(s)$$

c. The open-loop and closed-loop time constants are

$$T_{\text{open}} = T, \qquad T_{\text{closed}} = \frac{T}{1+Kb}$$

The closed-loop system responds $1 + Kb$ times faster than the open-loop system.

d. The static gains from disturbance to output are

$$G_{\text{yv,open}}(0) = 1, \qquad G_{\text{yv,closed}}(0) = \frac{1}{1+Kb}$$

The influence of disturbances is reduced a factor $1+Kb$ by the controller.

e. The static gains from input to output are

$$G_{\text{open}}(0) = b, \qquad G_{\text{closed}}(0) = \frac{Kb}{1+Kb}$$

The variation of gain in open loop is

$$\frac{G_{\text{open}}(0)_{\max}}{G_{\text{open}}(0)_{\min}} = \frac{5}{0.5} = 10$$

The variation in closed loop is

$$\frac{G_{\text{closed}}(0)_{\max}}{G_{\text{closed}}(0)_{\min}} = \frac{\frac{5K}{1+5K}}{\frac{0.5K}{1+0.5K}} = 10\frac{1+0.5K}{1+5K} \in (1, 10)$$

This shows that if we use small values of K, the gain may vary a factor 10. If we increase K the variation in closed loop decreases. If $K \to \infty$ the variation disappears, but then there will be problems with measurement noise and large control signals, which should be avoided.

7.5 (Extra)a. The transfer function from v to e is

$$G_{ev}(s) = \frac{-G_p(s)}{1 + KG_p(s)}$$

and the Laplace transform of $v(t)$ is $V(s) = 1/s$. The final value theorem then gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sG_{ev}(s)V(s) = G_{ev}(0) = -\frac{1}{2+K}$$

under the condition that $G_{ev}(s)$ is asymptotically stable.

b. The characteristic equation is $1 + KG_p(s) = 0$:

$$s^3 + 6s^2 + (12-K)s + (8+4K) = 0$$

Stability conditions:

$$\left.\begin{array}{ll} 12 - K > 0 & \Rightarrow K < 12 \\ 8 + 4K > 0 & \Rightarrow K > -2 \\ 6(12-K) > 8 + 4K & \Rightarrow K < 6.4 \end{array}\right\} \Rightarrow \quad -2 < K < 6.4$$

To get $|e(\infty)| = 0.1$ we would like to choose $K = 8$. But it is not possible to choose $K > 6.4$ since the system will then be unstable. Note that, in the case of a step input signal, the final value theorem can only be used if the system is asymptotically stable!

7.6 (Extra)a. Using the control law

$$u = K_1 d + K_2 r,$$

we get the output

$$y = (G_1K_1 + G_2)d + G_1K_2r$$

It is seen that we can choose $K_1 = -\frac{G_2}{G_1}$ and $K_2 = \frac{1}{G_1}$ to eliminate the influence of d, giving $y = r$.

b. Assuming $G_1 = 0.1$ and $G_2 = 1$ gives the controller

$$u = -10d + 10r$$

Now, if the real system is changed to $G_1 = -0.1$ and $G_2 = 1.1$, the resulting output is

$$y = (G_1K_1 + G_2)d + G_1K_2r = 2.1d - r$$

which is very far from the desired relation. The disturbance is amplified and the output has the wrong sign with respect to the reference value. The feedforward controller yields a system that is very sensitive to variations in the system parameters, especially G_1.

c. Using the control law

$$u = K_1y + K_2r$$

we get the output

$$y = G_1K_1y + G_1K_2r + G_2d$$

$$y = \frac{G_1K_2}{1 - G_1K_1}r + \frac{G_2}{1 - G_1K_1}d.$$

In order to reduce the influence of the disturbance d, we want to make $1 - G_1K_1$ big. For this purpose, choose for instance $K_1 = 1000$. Since we want y to equal r if no disturbances are present, we want to have

$$\frac{G_1K_2}{1 - G_1K_1} = 1$$

meaning that we should choose $K_2 = 1010$. This choice of controller parameters gives the system output

$$y = r + 0.01d$$

That is, a unit gain from r to y and a significant attenuation of the disturbance.

If the system parameters are changed to $G_1 = -0.1$ and $G_2 = 1.1$ the controlled system becomes

$$y = 1.02r - 0.01d.$$

As we can see, the system is not significantly changed. The relation between y and r is close to the requirement and the disturbance is still attenuated.

(In reality we would also have to consider the fact that large values of K can give large control signals and an increased sensitivity to measurement noise. In general it is recommended to use a combination of feedback and feedforward if possible.)

Solutions to Exercise 8. Control Systems Design

8.1 a. The desired characteristic polynomial is

$$(s+2)^2 = s^2 + 4s + 4$$

Comparing to $s^2 + 2\zeta\omega + \omega^2$, we see that $\omega = 2$ and $\zeta = 1$.

b. The closed-loop system is

$$G_{cl}(s) = \frac{G_p(s)G_r(s)}{1 + G_p(s)G_r(s)} = \frac{Ks + \frac{K}{T_i}}{s^2 + (K+1)s + \frac{K}{T_i}}$$

Identification with the desired characteristic polynomial $s^2 + 4s + 4$ gives the controller parameters $K = 3$ and $T_i = 0.75$.

8.2 —

8.3 —

8.4 Step response method: With the customary notion we obtain $a = 0.3$ and $b = 0.8$. The controller parameters become $K = 1.2/a = 4$, $T_i = 2b = 1.6$ and $T_d = b/2 = 0.4$.

Frequency method: The Nyquist curve intersects the negative real axis in $-1/3$ for $\omega = 1$ rad/s, which yields $T_o = 2\pi/\omega = 2\pi$ and $K_c = 3$. The controller parameters become $K = 0.6K_c = 1.8$, $T_i = T_o/2 = 3.1$ and $T_d = T_o/8 = 0.8$.

8.5 (Extra) The phase curve crosses $-180°$ when

$$\arg G(i\omega_o) = -3\arctan(\omega_o) = -\pi \quad \Rightarrow$$

$$\omega_o = \tan(\frac{\pi}{3}) = \sqrt{3} = 1.732$$

At this frequency, the gain is

$$|G(i\omega_o)| = (1 + \omega_o^2)^{-3/2} = 1/8 = 0.125$$

This yields $K_c = 8$ and the period $T_o = 2\pi/\omega_o = 3.6$.

The controller parameters become $K = 0.6K_c = 4.8$, $T_i = T_o/2 = 1.8$ and $T_d = T_o/8 = 0.45$.

Solutions to Exercise 9. Analysis in the Frequency Domain

9.1 a. The frequency function is

$$G(i\omega) = \frac{1}{1 + i\omega T}$$

with amplitude function

$$|G(i\omega)| = \frac{1}{\sqrt{1 + (\omega T)^2}}$$

and phase function

$$\arg G(i\omega) = -\arctan \omega T$$

b. The input is given by

$$u(t) = 5\sin(\omega_1 t) + 8\sin(\omega_2 t)$$

where $\omega_1 = \frac{2\pi}{24}$ rad/hour and $\omega_2 = \frac{2\pi}{24 \cdot 365}$ rad/hour. Since the system is linear, each component can be analyzed separately.

Over one day, the amplitude of the temperature variation will be

$$5|G(i\omega_1)| = 0.0038$$

which is not noticeable at all.

Over one year, the amplitude of the temperature variation will be

$$8|G(i\omega_2)| = 2.1$$

which is clearly noticable. The phase is

$$\arg G(i\omega_2) = -1.3$$

which means that temperature in the cave will lag by $\frac{0.96}{2\pi} = 0.21$ years ≈ 2.5 months compared to the outdoor temperature.

9.2 a. The output is given by

$$y(t) = |G(3i)| \sin\Big(3t + \arg G(3i)\Big)$$

where

$$|G(i\omega)| = \frac{0.01\sqrt{1 + 100\omega^2}}{\sqrt{1 + \omega^2}\sqrt{1 + 0.01\omega^2}}$$

and

$$\arg G(i\omega) = \arctan 10\omega - \arctan \omega - \arctan 0.1\omega$$

For $\omega = 3$ one obtains $|G(i\omega)| = 0.0909$ and $\arg G(i\omega) = -0.003$ which gives

$$y(t) = 0.0909 \sin(3t - 0.003)$$

b. Reading from the plot yields $|G(3i)| \approx 0.09$ and $\arg G(3i) \approx 0$. We obtain

$$y(t) = 0.09 \sin 3t$$

9.3 The closed-loop system is stable for those $K > 0$ that makes the Nyquist curve not encircle the point -1.

a. $K < 2$

b. $K < 1/1.5 = 2/3$

c. $K < 1/1.5 = 2/3$

d. $K < 1/(2/3) = 1.5$

9.4 a. The amplitude margin A_m tells how much the amplitude can be increased until the system becomes unstable. It can be calculated from

$$\begin{aligned} \arg G(i\omega_o) &= -180° \\ |G(iw_o)|A_m &= 1 \end{aligned}$$

From the Bode plot it can be seen that $\omega_o \approx 2.4$ rad/s and that $|G(i\omega_o)| \approx 0.6$, see Figure 9.1.

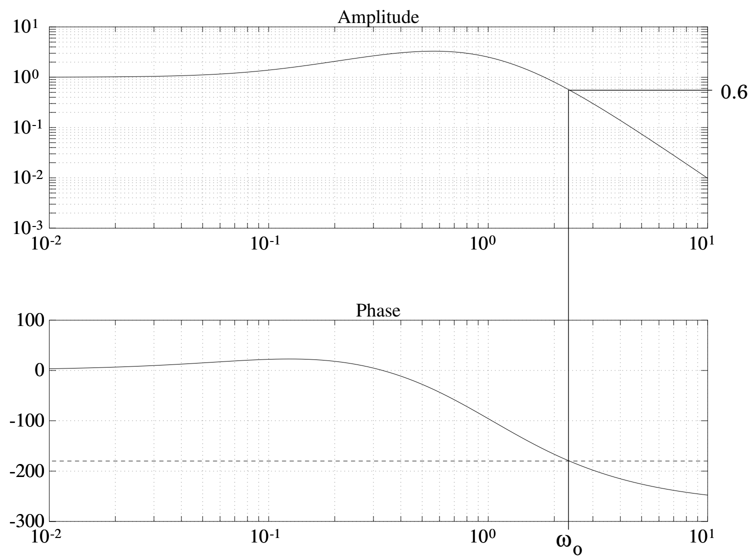


Figure 9.1 Bode diagram for the system $G(s)$ in Problem 9.4.

Also from the Nyquist plot it can be seen that $|G(i\omega_o)| \approx 0.6$, see Figure 9.2.

This means that the amplitude margin is

$$A_m = \frac{1}{0.6} = 1.67$$

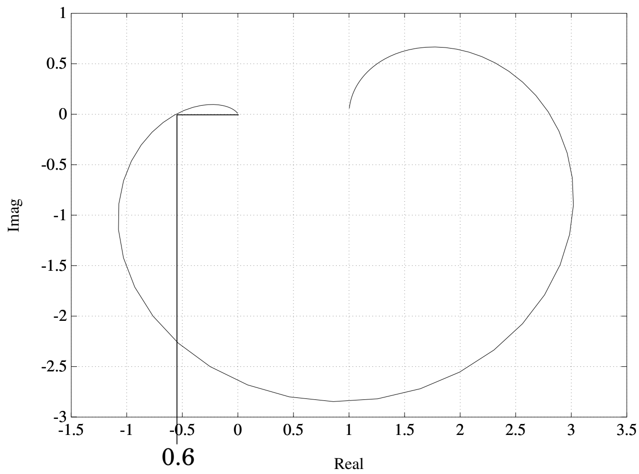


Figure 9.2 Nyquist diagram for the system $G(s)$ in Problem 9.4.

b. The phase margin φ_m tells how much the phase can be decreased before the system becomes unstable. It can be calculated from

$$|G(i\omega_c)| = 1$$
$$\arg G(i\omega_c) - \varphi_m = -180^\circ$$

From the Bode plot it can be seen that $\omega_c \approx 1.77$ and that $\arg G(i\omega_c) \approx -150^\circ$, see Figure 9.3.

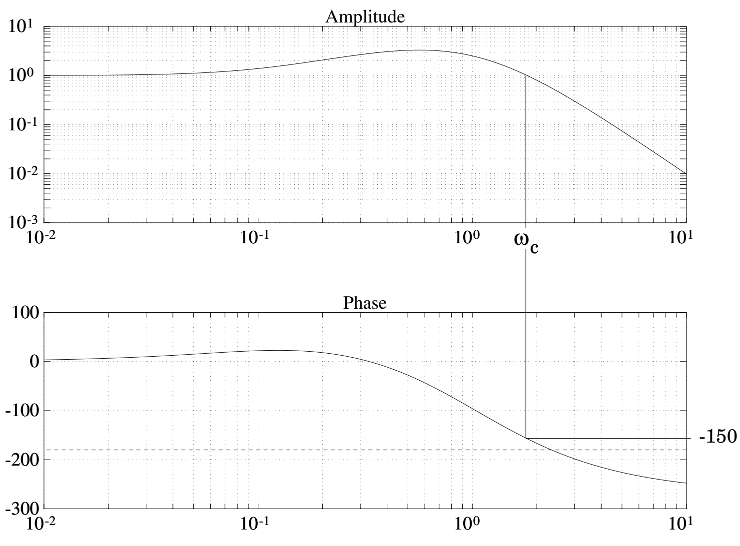


Figure 9.3 Bode diagram for the system $G(s)$ in Problem 9.4.

Also from the Nyquist plot it can be seen that $\arg G(i\omega_c) \approx -150^\circ$, see Figure 9.4.

This means that the phase margin is

$$\varphi_m = 180^\circ - 150^\circ = 30^\circ$$

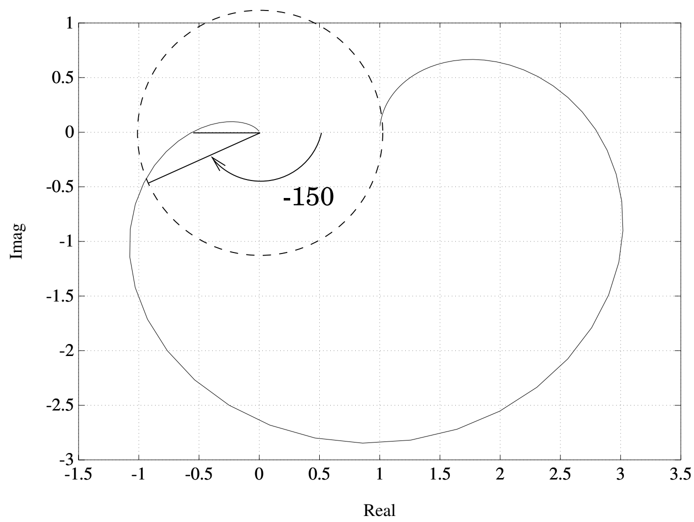


Figure 9.4 Nyquist diagram for the system $G(s)$ in Problem 9.4.

9.5 When the input signal $u(t)$ is a sinusoidal, the output signal $y(t)$ also becomes a sinusoidal (after a transient). The relation between the input signal and the output signal is given by

$$u(t) = \sin(\omega t)$$
$$y(t) = |G(i\omega)| \sin(\omega t + \varphi)$$

The amplitude margin A_m can be determined from the third frequency response. In this response, the phase $\varphi = -180°$ and the amplitude $|G(i\omega)| = 0.5$. The amplitude margin is given by

$$A_m = \frac{1}{0.5} = 2.0$$

The phase margin φ_m can be determined from the second frequency response. In this response, the amplitude is unchanged, $|G(i\omega)| = 1.0$, and the output y is delayed some time compared to the input u. Measurements in the figure give that the period of the input is approximately 5.5 seconds. The delay from a minimum in u to a minimum in y is roughly 2.5 seconds. This gives a phase lag of $\varphi = \frac{2.5}{5.5} \cdot 360 \approx 160°$. The phase margin is

$$\varphi_m \approx 180° - 160° \approx 20°$$

9.6 (Extra) The system is stable for

$$0 < K < \frac{1}{3.5} \quad \Leftrightarrow \quad 0 < K < 0.29$$

as well as for

$$1 < K < \frac{1}{0.5} \quad \Leftrightarrow \quad 1 < K < 2$$

Solutions to Exercise 10. Control Structures

10.1 Calculation of the the transfer function from $v(t) = T_y(t)$ to $y(t) = T_h(t)$ yields

$$G_{vy}(s) = \frac{H(s)G_2(s) + G_2(s)G_1(s)G_{ff}(s)}{1 + G_1(s)G_2(s)G_r(s)}$$

To eliminate the disturbance we need to choose $G_{ff}(s)$ so that

$$H(s) + G_1(s)G_{ff}(s) = 0$$

Hence

$$G_{ff}(s) = -\frac{H(s)}{G_1(s)} = -\frac{s+1}{s+0.5}$$

10.2a. The open inner loop $G_{valve}(s) = \frac{2}{s+2}$ has time constant $T = 1/2$. The closed inner loop

$$G_{in}(s) = \frac{K_1 G_{valve}}{1 + K_1 G_{valve}} = \frac{2K_1}{s + 2 + 2K_1}$$

has time constant $T = 1/(2 + 2K_1) = 1/10$, if we choose $K_1 = 4$, which makes the closed inner loop five times faster.

b. With the approximation $G_{in}(s) \approx G_{in}(0) = 8/10$ the outer loop becomes

$$G_{out}(s) = \frac{G_{PI}G_{tank}G_{in}(0)}{1 + G_{PI}G_{tank}G_{in}(0)} = \frac{(K_2 s + \frac{K_2}{T_I})\frac{8}{10}}{s^2 + K_2\frac{8}{10}s + \frac{K_2}{T_I}\frac{8}{10}}$$

The requirement that the outer loop should be ten times slower than the inner loop could be translated to $\omega = 1$. If we then match coefficients as usual in pole placement design we get $\omega^2 = \frac{K_2}{T_I}\frac{8}{10} = 1$. We also have to choose an appropriate damping coefficient ζ. Choose e.g. $K_2 = 2.5$ and $T_I = 2$ to get a double pole at -1, which means we have used the damping coefficient $\zeta = 1$.

Without the approximation we have the characteristic polynomial $s^3 + 10s^2 + 20s + 10$, which is stable.

c. Choose the feed forward such that $G_{FF}G_{in} = 1$. However, $1/G_{in}(s) = \frac{1}{8}s + \frac{10}{8}$ includes taking the derivative of the disturbance. It can be very difficult and introduce a lot of noise when using a derivate of a measured signal, so we use the inner loop approximation instead and choose $G_{FF} = 1/G_{in}(0) = 1/0.8 = 1.25$. At least the steady state error from a step disturbance is then eliminated.

d. The process flow sheet is given in Figure 10.1.

10.3a. The block diagram is shown in Figure 10.2.

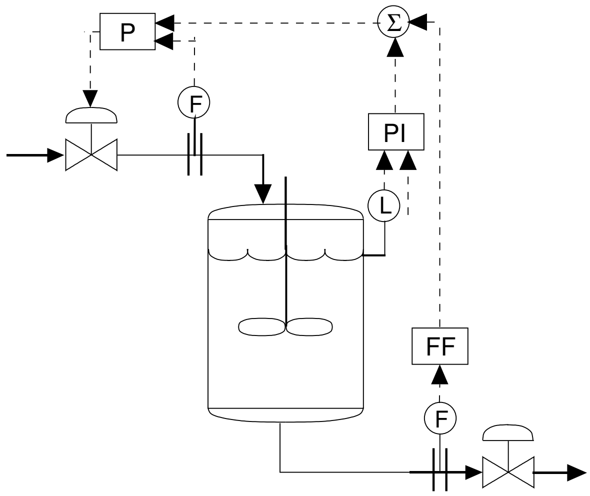


Figure 10.1 The control system in Problem 10.2.

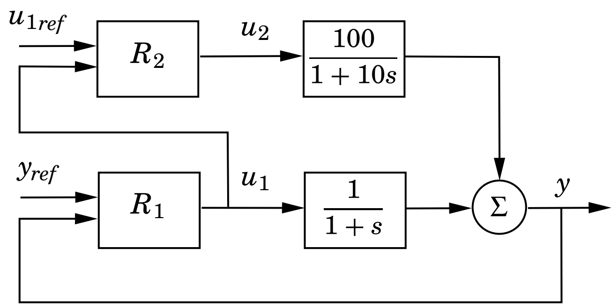


Figure 10.2 Mid-range control of a steam flow.

b. The purpose of R_1 is to handle small disturbances in the steam flow quickly. This is best done with the P-controller. The purpose of R_2 is to slowly adjust the large steam flow such that R_1 operates within the center of its range and does not saturate. This is best done with the I-controller.

c. With $P_1 = \frac{1}{1+s}$ and $P_2 = \frac{100}{1+10s}$ the closed-loop transfer function can calculated as

$$G_{cl} = \frac{R_1(P_1 + R_2P_2)}{1 + R_1(P_1 + R_2P_2)} == \frac{\frac{1}{1+s} + \frac{1}{100s}\frac{100}{1+10s}}{1 + \frac{1}{1+s} + \frac{1}{100s}\frac{100}{1+10s}} = \frac{10s^2 + 2s + 1}{10s^3 + 21s^2 + 3s + 1}$$

The stationary gain of the closed-loop system is $G_{cl}(0) = 1$, which is exactly what we want.

10.4 (Extra) A block diagram for the system is shown in figure 10.3. Mass balance for the tank yields

$$A\frac{dh}{dt} = x(t) - v(t)$$

Laplace transformation gives ($A = 1\ m^2$)

$$H(s) = \frac{1}{s}(X(s) - V(s))$$

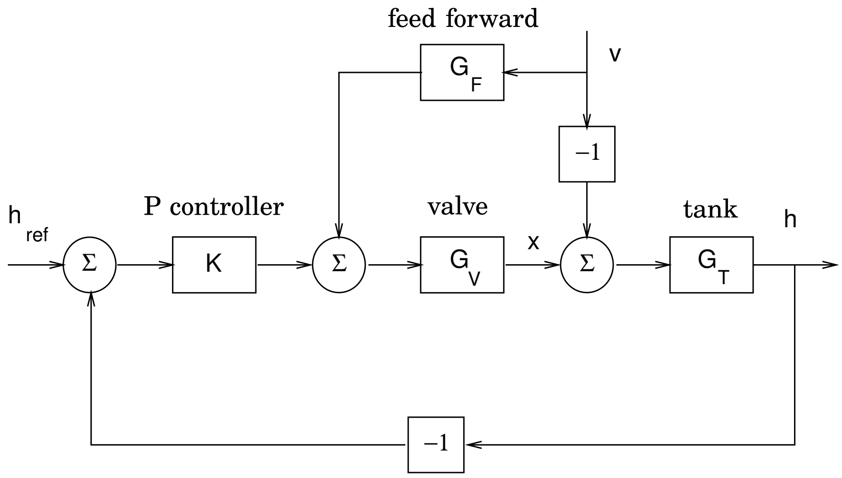


Figure 10.3 Block diagram of the level controlling system in Problem 10.4 (Extra).

The transfer function of the tank is thus

$$G_T(s) = \frac{1}{s}$$

a. The closed loop transfer function becomes

$$G(s) = \frac{G_T G_V K}{1 + G_T G_V K} = \frac{K}{0.5s^2 + s + K}$$

The characteristic polynomial is hence

$$s^2 + 2s + 2K$$

The desired characteristic polynomial is

$$(s + \omega)^2 = s^2 + 2\omega s + \omega^2$$

Identification of coefficients yields

$$\begin{cases} \omega = 1 \\ K = \dfrac{1}{2} \end{cases}$$

The transfer function form $v(t)$ to $h(t)$ is given by

$$H(s) = -\frac{G_T}{1 + G_T G_V K} V(s) = -\frac{1 + 0.5s}{s(1 + 0.5s) + K} V(s)$$

If $v(t)$ is a step of amplitude 0.1 we obtain $V(s) = 0.1/s$. The final value theorem gives

$$h(\infty) = \lim_{s \to 0} sH(s) = -\frac{0.1}{K}$$

given that the limit exists and that the final value theorem is applicable.

b. A PI controller has the transfer function

$$G_R(s) = K(1 + \frac{1}{sT_i})$$

The closed loop transfer function becomes

$$G(s) = \frac{G_T G_V G_R}{1 + G_T G_V G_R} = \frac{K(1 + sT_i)}{s(1 + 0.5s)sT_i + K(1 + sT_i)}$$

The characteristic polynomial becomes

$$s^3 + 2s^2 + 2Ks + \frac{2K}{T_i}$$

The desired characteristic polynomial is

$$(s + \omega)^3 = s^3 + 3\omega s^2 + 3\omega^2 s + \omega^3$$

Identification of coefficients yields

$$\begin{cases} \omega = \dfrac{2}{3} \\ K = \dfrac{2}{3} \\ T_i = \dfrac{9}{2} \end{cases}$$

c. The relation between the flow disturbance v and the level h is given by

$$H(s) = \frac{G_T(G_V G_F - 1)}{1 + G_T G_V G_R} V(s)$$

To eliminate the influence of v, we choose

$$G_F(s) = \frac{1}{G_V} = 1 + 0.5s$$

A. MATLAB help

Some useful commands from Control System Toolbox

```
General.
  ltimodels   - Detailed help on the various types of LTI models.

Creating linear models.
  tf          - Create transfer function models.
  zpk         - Create zero/pole/gain models.
  ss          - Create state-space models.
  set         - Set/modify properties of LTI models.

Data extraction.
  tfdata      - Extract numerator(s) and denominator(s).
  zpkdata     - Extract zero/pole/gain data.
  ssdata      - Extract state-space matrices.
  get         - Access values of LTI model properties.

Conversions.
  tf          - Conversion to transfer function.
  zpk         - Conversion to zero/pole/gain.
  ss          - Conversion to state space.

System interconnections.
  parallel    - Generalized parallel connection (see also overloaded +)
  series      - Generalized series connection (see also overloaded *).
  feedback    - Feedback connection of two systems.

Model dynamics.
  dcgain      - D.C. (low frequency) gain.
  pole, eig   - System poles.
  zero        - System (transmission) zeros.
  pzmap       - Pole-zero map.

Time-domain analysis.
  step        - Step response.
  impulse     - Impulse response.

Frequency-domain analysis.
  bode        - Bode diagrams of the frequency response.
  nyquist     - Nyquist plot.
  margin      - Gain and phase margins.

Overloaded arithmetic operations.
  + and -     - Add and subtract systems (parallel connection).
  *           - Multiply systems (series connection).
```

Some useful commands from Simulink

```
Linearization and trimming.
  linmod           - Extract linear model from continuous-time system.
  trim             - Find steady-state operating point.

Simulation.
  sim              - Simulate a Simulink model from the command line.
  simset           - Define options to SIM Options structure.
```