Impulse and step response
Connection between transfer function and step response
Nonlinear systems

Reading: Systems Engineering and Process Control: 5.1–5.3
LTI systems – repetition (L3–L4)

State-space model

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

System response:

\[ x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \]
\[ y(t) = Cx(t) + Du(t) \]

Stability:
Decided by eigenvalues of \( A \)

Input-output model

\[ U(s) \]
\[ G(s) \]
\[ Y(s) \]

System response:

\[ Y(s) = G(s)U(s) \]

Stability:
Decided by poles to \( G(s) \)
Suppose that the system is in equilibrium
How does output react to input impulse (Dirac function)?
Impulse response for linear systems

1. Laplace transform input: \( U(s) = 1 \)
2. Output becomes:

\[
H(s) = G(s)U(s) = G(s)
\]

3. Inverse transform gives impulse response:

\[
h(t) = \mathcal{L}^{-1}\{G(s)\}
\]

\( h(t) \) also called weighting function
Impulse response for linear systems

For a system on state-space for the impulse response becomes:

\[ h(t) = Ce^{At}B + D\delta(t) \]

Stability notions (again):

- \( h(t) \) limited (except maybe at \( t = 0 \) ) \( \iff \) stable system
- \( h(t) \to 0 \) \( \iff \) asymptotically stable system
- \( h(t) \) unlimited \( \iff \) unstable system
Step response

- Suppose system in equilibrium
- How does the output change after step in input?
Step response for linear systems

1. Laplace transform input: \( U(s) = \frac{1}{s} \)

2. Output becomes:

\[
Y(s) = G(s)U(s) = G(s)\frac{1}{s}
\]

3. Inverse transformation gives step response:

\[
y(t) = \mathcal{L}^{-1}\left\{ G(s)\frac{1}{s} \right\} = \int_0^t h(\tau)d\tau
\]

(The step response is the integral of the impulse response)
Static gain

- Step response end value is called **static gain** of system.
- Can be computed using the end value theorem:

\[ Y(s) = G(s) \frac{1}{s} \]

\[ \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s) \frac{1}{s} = G(0) \]

- Note: Step response end value exists only for asymptotically stable systems!
Example: CSTR

Transfer function from $c_{R,in}$ to $c_P$:

$$G(s) = \frac{\frac{q}{V}k}{(s + \frac{q}{V} + k)(s + \frac{q}{V})}$$

Static gain:

$$G(0) = \frac{\frac{q}{V}k}{(\frac{q}{V} + k)\frac{q}{V}} = \frac{1}{\frac{q}{kV} + 1}$$

Interpretation: If $c_{R,in}$ increases with 1, $c_P$ increases with $\frac{1}{\frac{q}{kV} + 1}$ at equilibrium
Connection between transfer fcn and step response

System type:

- Integrator
- First order system
- Second order systems with real poles
- Second order systems with complex poles
- Systems with one zero
- Systems with time delays
Integrating systems

\[ G(s) = \frac{K}{s} \]

Example: Tank without free outflow:

- Cross-sectional area: \( A \)

Transfer function from \( q_{\text{in}} \) to \( h \):

\[ G(s) = \frac{1/A}{s} \]
Integrating systems

- Pole:
  \[ s = 0 \]

- Step response:
  \[ Y(s) = G(s) \frac{1}{s} = \frac{K}{s^2} \]
  \[ y(t) = Kt \]

- No end value, since system is not asymptotically stable
Integrating systems

\( K = 1: \)

**Singularitetsdiagram**

**Stegsvar**
1st order systems

\[ G(s) = \frac{K}{1 + sT}, \quad T > 0 \]

Example: Temperature dynamics in a tank:

Transfer function from \( \theta_0 \) to \( \theta_1 \):

\[ G(s) = \frac{1}{1 + s\frac{V}{q}} \]
1:a order systems

- **Pole:**
  
  \[ s = -\frac{1}{T} \]

- **Step response:**
  
  \[ Y(s) = G(s) \frac{1}{s} = \frac{K}{s(1 + sT)} \]
  
  \[ y(t) = K \left( 1 - e^{-t/T} \right) \]

- \( T \) is called **time constant** of the system
  
  \[ y(T) = (1 - e^{-1})K \approx 0.63K \]
$K = 1$:

- Step response speed decided by distance from pole to origin
2nd order systems with real poles

\[ G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}, \quad T_1, T_2 > 0 \]

Example: Temperature dynamics in coupled tanks:

Transfer function from \( \theta_0 \) to \( \theta_2 \):

\[ G(s) = \frac{1}{(1 + s \frac{V_1}{q})(1 + s \frac{V_2}{q})} \]
2:a order systems with real poles

- Poles:
  \[ s = -1/T_1, \quad s = -1/T_2 \]

- Step response:
  \[
  Y(s) = G(s) \frac{1}{s} = \frac{K}{s(1 + sT_1)(1 + sT_2)}
  \]

  \[
  y(t) = \begin{cases} 
  K \left(1 - \frac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2}\right), & T_1 \neq T_2 \\
  K \left(1 - e^{-t/T} - \frac{t}{T} e^{-t/T}\right), & T_1 = T_2 = T
  \end{cases}
  \]

- Two time constants: \( T_1, T_2 \)
2nd order systems with real poles

\( K = 1: \)

- Two poles gives softer and slower response than single pole
  - equivalent time constant: \( T_{eq} = T_1 + T_2 \)
  - If \( T_1 \gg T_2 \) system behaves essentially as 1st order system with time constant \( T_1 \)
2nd order systems with complex poles

\[ G(s) = \frac{K \omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2}, \quad \omega_0 > 0, \ 0 < \zeta < 1 \]

- \( \omega_0 \) = undamped frequency
- \( \zeta \) = relative damping

Example: Position dynamics for mechanical system

Transfer function from \( F \) to \( z \):

\[ G(s) = \frac{1}{m s^2 + \frac{d}{m} s + \frac{k}{m}} \]

Complex poles if \( d < 2\sqrt{km} \)
2nd order systems with complex poles

Poles:

\[ s = -\zeta \omega_0 \pm i \sqrt{1 - \zeta^2} \omega_0 \]

\[ \zeta = \cos \varphi \]

Step response:

\[ y(t) = K \left( 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_0 t} \sin \left( \omega_0 \sqrt{1 - \zeta^2} t + \arccos \zeta \right) \right) \]
2nd order systems with complex poles

\[ K = 1: \]

- System speed decided by distance from poles to the origin
2nd order systems with complex poles

\( K = 1: \)

System damping decided by angle of the poles
Suppose the system is given by

\[(1 + T_z s)G_0(s)\]

Zero in \(s = -\frac{1}{T_z}\)

Step response:

\[y(t) = \mathcal{L}^{-1}\left\{G_0(s)\frac{1}{s}\right\} + T_z\mathcal{L}^{-1}\{G_0(s)\}\]

Weighted sum of impulse response and step response for \(G_0(s)\)

Big impact if zero close to the origin \((T_z\) large)
2nd order systems with zeros

Example: \( G(s) = \frac{1 + sT_z}{(1 + 2s)^2} \)

Singularity Chart

Step Response

Dashed step response for \( G_0(s) = \frac{1}{(1 + 2s)^2} \)

- Zeros affect initial response
- R.h.p. zeros gives inverse response behavior initially
Systems with time delay

Suppose the system is given by:

\[ G(s) = G_0(s)e^{-sL}, \quad L > 0 \]

Step response for part without delay \( G_0(s) \):

\[ y_0(t) = \mathcal{L}^{-1} \left\{ G_0(s) \frac{1}{s} \right\} \]

Step response with time delay:

\[ y(t) = y_0(t - L) \]

\( e^{-sL} \) cannot be interpreted with (finitely many) poles and zeros.)
Interpretation of poles and zeros

Poles

- Depends only on $A$-matrix, e.g., on system inner dynamics
- Decides system:
  - stability
  - speed
  - damping

Zeros

- Harder to interpret
- Depends on how inputs and outputs are connected to system (i.e., depends on $B$, $C$, and $D$ matrices)
- A zero in $s = a$ cancels the signal $e^{at}$
- Influences mostly the initial step response behavior
Processes that are difficult to control

Processes with:

- Poles in right half-plane (unstable)
  - The bigger the real part \((> 0)\) the harder to control
- Zeros in right half-plane (reversed response initially)
  - The smaller real part \((> 0)\) the harder to control
- Time delays
  - The longer time delay, the harder to control
Processes that are impossible to control

- Systems with poles and zeros in right half-plane \((a, b > 0)\):

  \[
  G(s) = \frac{Q(s)(s - a)}{P(s)(s - b)}
  \]

- If \(a = b\): impossible to control
- If \(a \leq 3b\): impossible to control in practice
Examples

Bicycle with back wheel steering
\( a/b \approx 0.7 \) at 1 m/s

X29, \( a/b \approx 4.33 \)
Nonlinear systems

Different kinds of nonlinearities:

- **Nonlinearities in actuators and sensors, e.g.,:**
  - upper and lower limits on actuators and sensors
  - pumps and valves with nonlinear characteristics
  - friction and dead zones
  - nonlinear sensors for temperature, flow, concentration

- **Nonlinear dynamics in the process, e.g.,:**
  - level dependent outflow speed in a tank
  - temperature dependent reaction speed in reactor
  - population dependent rate of growth

- **Nonlinearities in the controller, e.g.,:**
  - on/off control
Example: Valve and pump characteristics

Methods to compensate for nonlinearity:

- Compensate with table/mathematical function
- Feedback around static nonlinearity (better and more robust)
Example: pH control

Want to control pH but measures concentration:

- Can be compensated for with table/mathematical function
Example: Logistic growth model

\[ \frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) \]

\( x = \) population, \( r = \) net growth rate, \( k = \) carrying capacity
Example: Hares and Lynxes

\[
\frac{dH}{dt} = rH \left(1 - \frac{H}{k}\right) - \frac{aHL}{c + H}, \quad H \geq 0,
\]

\[
\frac{dL}{dt} = b \frac{aHL}{c + H} - dL, \quad L \geq 0
\]
Linear vs nonlinear systems

Linear systems

- can equivalently be described with linear differential equation, state-space model, transfer function, impulse response, or step response
- are always in equilibrium at \((x, u) = 0\)
- global analysis – poles/zeros decide stability globally

Nonlinear systems

- described by nonlinear differential equation/state-space model
- can have many equilibria (stable/unstable) and limit cycles
- simulation, local analysis using linearization