



LUNDS
UNIVERSITET

Institutionen för
REGLERTEKNIK

Automatic Control, Basic Course (FRTF05)

Exam December 18, 2018, 13:00-18:00
in room F102 of New Main Building, BUAA

Points and grades

All answers must include a clear motivation. The maximal number of points is 25. The maximal number of points is specified for each subproblem, and the grades are shown below.

3: minimum 12 points

4: minimum 17 points

5: minimum 22 points

Accepted aids

Mathematical collections of formulae (e.g. TEFYMA); 'Collections of formulae in automatic control'; calculator that is not programmed in advance.

Results

The results will be posted on the course home page and the graded exam will be displayed on Tuesday January 8, in lab C, Dept of Automatic Control. at 12.30-13.00. Thereafter, exams will be archived at the Automatic Control Department in Lund and at BUAA, Beijing, respectively.

Solutions

1. Study the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -1 & 3 \\ a & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= [0 \quad 1] x\end{aligned}$$

- a. How many inputs, outputs and states does this system have? (1 p)
- b. For which $a \in \mathbb{R}$ is the system not controllable? (1.5 p)
- c. For this a , which states can not be reached from the origin with any control signal? (1.5 p)

Solution

- a. One input, one output and two states.
- b. The system is not controllable if the controllability matrix $W_c = [B \ AB]$ is not of full rank. This is the same as the determinant of the matrix is equal to zero.

$$\begin{aligned}\det([B \ AB]) &= \det\left(\begin{bmatrix} 1 & 2 \\ 1 & a-2 \end{bmatrix}\right) \\ &= a - 2 - 2 = 0 \\ a &= 4\end{aligned}$$

- c. The states we can reach are all linear combinations of the columns in the controllability matrix. For $a = 4$ we get

$$W_c = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

which tells us that the reachable states are on the form

$$x = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so any states not on this form cannot be reached, i.e. any state where $x_1 \neq x_2$.

2. Consider the system that you studied in the previous problem,

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -1 & 3 \\ a & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y &= [0 \quad 1] x.\end{aligned}$$

- a. Find a linear state feedback, $u = Lx$, where $L = [l_1, l_2] \in \mathbb{R}^{1 \times 2}$, such that the poles of the closed loop system are placed in $s = -1 \pm i$ for every a at which the system is controllable. (2.5 p)

- b. What happens with the feedback gain and the resulting control signals as the parameter a approaches a value for which the system is uncontrollable? (0.5 p)

Solution

- a. Assume a positive feedback interconnection $u = Lx$, where $L = [l_1, l_2] \in \mathbb{R}^{1 \times 2}$. The closed loop dynamics can be written,

$$\dot{x} = Ax + Bu = (A + BL)x, \quad (1)$$

with a corresponding characteristic polynomial

$$p(s) = \det(sI - (A + BL)) \quad (2)$$

$$= \det \left(\begin{bmatrix} s + 1 - l_1 & -3 - l_2 \\ -a - k_1 & s + 2 - l_2 \end{bmatrix} \right) \quad (3)$$

$$= (s + 1 - l_1)(s + 2 - l_2) - (-3 - l_2)(-a - l_1) \quad (4)$$

$$= s^2 + (3 - l_1 - l_2)s + 2 - (1 + a)l_2 - 5l_1 - 3a. \quad (5)$$

In order to satisfy the specifications, the closed loop polynomial should be

$$p_m(s) = (s + 1 + i)(s + 1 - i) = s^2 + 2s + 2. \quad (6)$$

To choose L , simply let $p(s) = p_m(s)$, whereby

$$\begin{cases} s^2 : 1 = 1 \\ s^1 : 2 = 3 - l_1 - l_2 \\ s^0 : 2 = 2 - (1 + a)l_2 - 5l_1 - 3a \end{cases} \quad (7)$$

which can be written as a linear system in k , and subsequently solved

$$\begin{bmatrix} -1 & -1 \\ -5 & -(1 + a) \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3a \end{bmatrix} \quad (8)$$

\Downarrow

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \frac{1}{a - 4} \begin{bmatrix} -(1 + a) & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3a \end{bmatrix} \quad (9)$$

\Downarrow

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \frac{1}{a - 4} \begin{bmatrix} 4a + 1 \\ -3a - 5 \end{bmatrix}. \quad (10)$$

- b. From the previous problem, we know that the system is uncontrollable at $a = 4$. And it is clear that we require more and more control effort to steer the system when $a \rightarrow 4$, as for any $x_1 \neq x_2, x \neq 0$, the control signal becomes $\lim_{a \rightarrow 4} |u| = \infty$.

3. Figure 1 depicts the Bode diagram of a third order process.

- a. Determine the output $y(t)$ of the system if the control signal is given by $u(t) = 2 \sin(3t)$ for $-\infty < t < \infty$. (2 p)

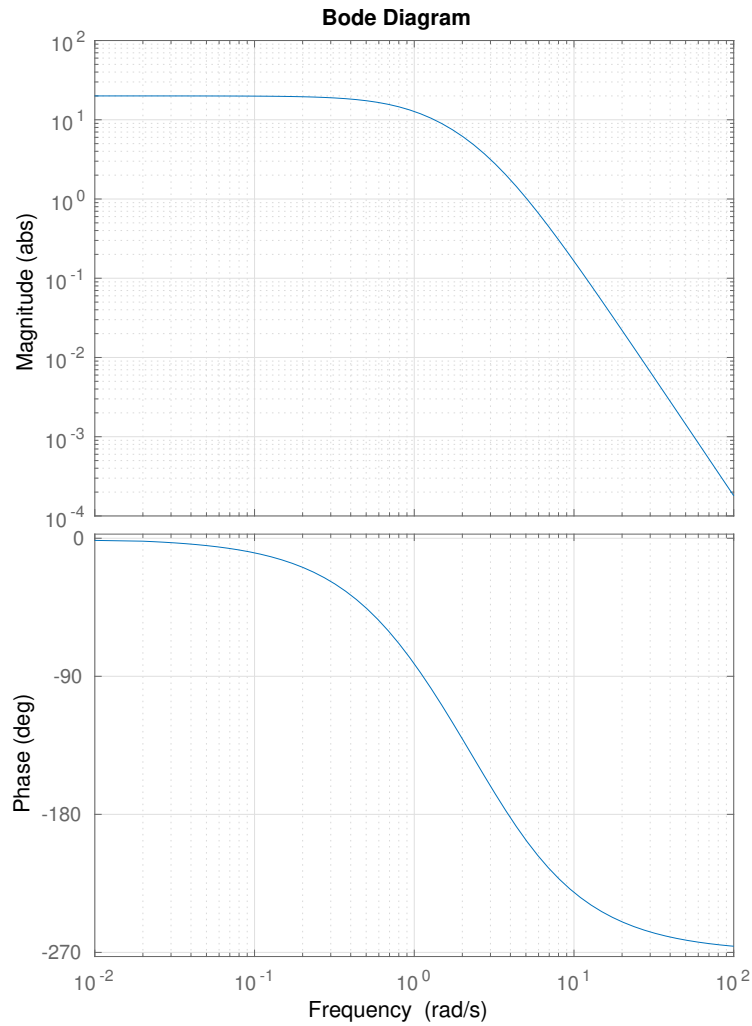


Figure 1: Bode diagram for Problem 3.

- b.** We close the loop by adding feedback and a P-controller with $K = 0.33$. What is the delay margin for the closed loop system? (2 p)
- c.** Create a closed loop by adding a PI-controller, and calibrate the system with Ziegler-Nichol's method. (2 p)

Solution

- a.** We have a control signal with an angular frequency of 3 rad/s. The Bode diagram gives us $|G(3i)| \approx 3$ and $\arg(G(3i)) \approx -160^\circ$ (exact values are $\sqrt{10}$ and $-180^\circ + \arctan(1/3)$). This results in the output

$$u(t) = 6 \sin(3t - 160\pi/180).$$

- b.** For a P-controller with $K = 0.33$ we want to find what ω_c that gives us $|KG(i\omega_c)| = 1$, i.e. the angular frequency that gives an amplitude of ~ 3 .

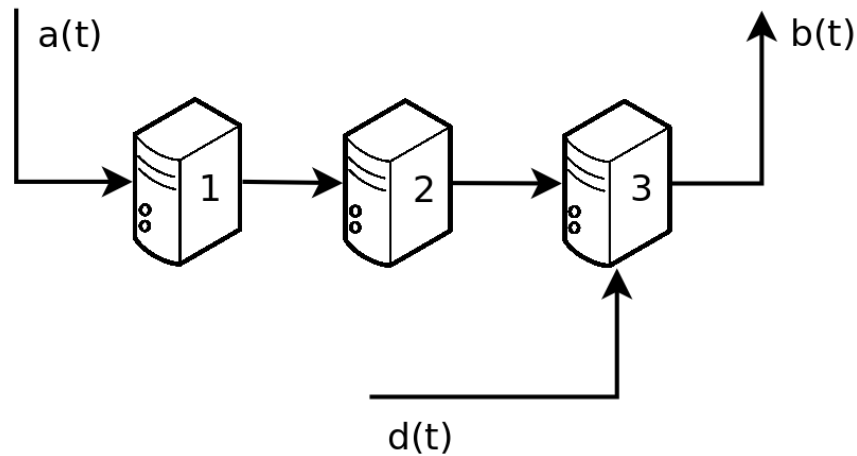
In the Bode diagram we can see that this happens around $\omega_c \approx 3$ rad/s and then we get $\phi_m = 180^\circ + \arg(G(i\omega_c)) \approx 20^\circ$.

$$L_m = \frac{\phi_m}{\omega_c} = \frac{20\pi}{3 \cdot 180} \approx 0.1164 \text{ s}$$

- c. First, we need to figure out what K_0 gives resonance during feedback and what the frequency of the resonance is. We find $\omega_0 \approx 4$ in the phase diagram and see that $|G(4i)| \approx 2$ at that point. This tells us that feedback with $K_0 = 0.5$ results in resonance (more exact values are 3.87 rad/s and 0.533). From the collection of formulae we get a PI of the following form:

$$K = 0.45K_0 \approx 0.2$$
$$T_i = \frac{T_0}{1.2} = \frac{2\pi}{1.2\omega_0} \approx 0.5$$

4. When designing applications for the cloud, it is usually recommended to split the code base into smaller self-contained *microservices*, which are then deployed on a cloud platform. Each microservice can be regarded as a virtual server. One such setup can be seen in Figure 2.



Figur 2: User requests $a(t)$ routed into server 1, which are further sent to server 2 and finally server 3, which routes packages back to the user. On server 3, high priority requests $d(t)$ can be routed from an outside server.

If the throughput is large, the dynamics of a server resembles that of a water tank with in- and outflow

$$\dot{q}(t) = p_{in}(t) - p_{out}(t),$$

where $q(t)$ is the average queue length and $p_{in}(t)$, $p_{out}(t)$ the average arrivals/departures. Commonly, the outflow from a server can be modeled as

$$p_{out}(t) = \mu \frac{q(t)}{q(t) + 1}$$

where μ is the inherent service rate of the server. For our system, $(\mu_1, \mu_2, \mu_3) = (5, 5, 3)$. Since the final server is the slowest, it is important that we keep its amount of packages limited to a set-point as not to overload the system. A controller $G_r(s)$ has been created that acts on $a(t)$ in order to keep the level on server 3 at the set point.

- a. Create a state-space model for the system, and linearize it around $u^0 = 1$. For now, assume that $d(t) = 0$. (2 p)
- b. By feed-back alone, the disturbances on server 3 can only be accounted for retroactively. This is bad, as it renders our system sensitive to large step disturbances which can overload the server.

To combat this, an additional feedforward controller can be added to the system using the known disturbance $d(t)$ as the input. Draw the block diagram of the entire system with the two controllers. Calculate the feedforward controller with respect to the linearized system such that the disturbance disappears. Is the controller implementable? If not, suggest at least one improvement to make it implementable. (2 p)

Solution

a. The state space form of the server system becomes

$$\dot{x}_1 = -\mu_1 \frac{x_1}{x_1 + 1} + u \quad (11)$$

$$\dot{x}_2 = -\mu_2 \frac{x_2}{x_2 + 1} + \mu_1 \frac{x_1}{x_1 + 1} \quad (12)$$

$$\dot{x}_3 = -\mu_3 \frac{x_3}{x_3 + 1} + \mu_2 \frac{x_2}{x_2 + 1} \quad (13)$$

$$y = x_3 \quad (14)$$

where $(\mu_1, \mu_2, \mu_3) = (5, 5, 3)$. Stationary point, given $u^0 = \alpha = 1$.

$$(x_1^0, x_2^0, x_3^0, u^0) = \left(\frac{\alpha}{\mu_1 - \alpha}, \frac{\alpha}{\mu_2 - \alpha}, \frac{\alpha}{\mu_3 - \alpha}, \alpha \right) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1 \right) \quad (15)$$

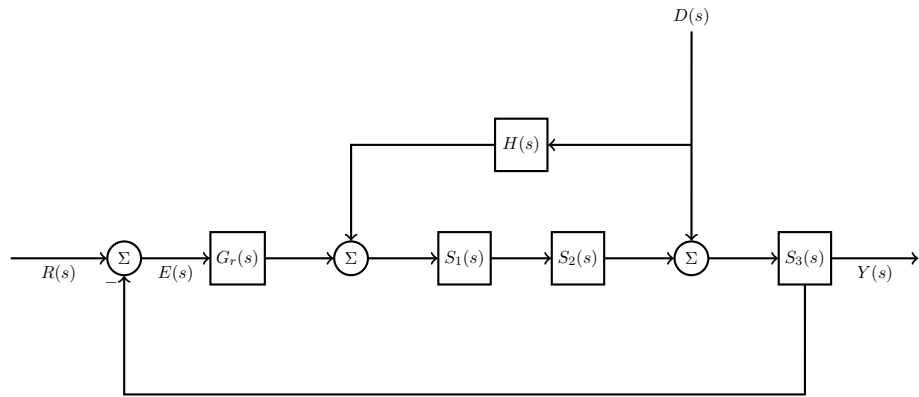
Linearize the system dynamics around $u^0 = 1$. Jacobian becomes

$$\begin{array}{cccc} \frac{df_1}{dx_1} = \frac{-\mu_1}{(x_1+1)^2} & \frac{df_1}{dx_2} = 0 & \frac{df_1}{dx_3} = 0 & \frac{df_1}{du} = 1 \\ \frac{df_2}{dx_1} = \frac{\mu_1}{(x_1+1)^2} & \frac{df_2}{dx_2} = \frac{-\mu_2}{(x_2+1)^2} & \frac{df_2}{dx_3} = 0 & \frac{df_2}{du} = 0 \\ \frac{df_3}{dx_1} = 0 & \frac{df_3}{dx_2} = \frac{\mu_2}{(x_2+1)^2} & \frac{df_3}{dx_3} = \frac{-\mu_3}{(x_3+1)^2} & \frac{df_3}{du} = 0 \end{array} \quad (16)$$

which yields the system

$$\Delta \dot{x} = \begin{pmatrix} -3.2 & 0 & 0 \\ 3.2 & -3.2 & 0 \\ 0 & 3.2 & -4/3 \end{pmatrix} \Delta x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Delta u \quad (17)$$

$$\Delta y = (0 \ 0 \ 1) \Delta x \quad (18)$$



Figur 3: Control system with disturbance and feed-forward for Problem 4.

b. The optimal feed-forward controller to remove disturbances is given by

$$H(s) = -\frac{1}{S_1(s)S_2(s)}. \quad (19)$$

In our linearized system, this becomes

$$H(s) = -\frac{1}{\frac{1}{s+3.2} \frac{3.2}{s+3.2}} = -\frac{(s+3.2)^2}{3.2} \quad (20)$$

The problem with this controller is that we differentiate the disturbance signal, making it sensitive to high frequencies. Two ways to make it practical is either by introducing a low-pass filter or relying on the static feedforward

$$H(s) = -\frac{3.2^2}{3.2} = -3.2 \quad (21)$$

instead.

5. Christmas is upon us and Santa Claus needs to plan the production of presents from his factories. It is important that the amount of produced gifts are not too few nor too many, or Santa might risk ruining Christmas or his tight budget. The amount of gifts produced can be controlled by tuning the production speed of the factories, the total output can be modeled as the following transfer function from $u(t)$ - *production speed*, to $y(t)$ - *millions of gifts produced*

$$G_p(s) = \frac{1}{s(s+1)}. \quad (22)$$

This model is far from perfect, the imperfections can be thought of as two disturbances, $d_1(t)$ affecting the control signal $u(t)$ and $d_2(t)$ affecting the output signal $y(t)$. Further, the actual production will need to follow a ramp reference $r(t) = \alpha t$, as peoples ability to postpone writing their wish lists makes it hard to produce the correct amount of gifts in one batch. To control the factories, Santa has utilized feedback with a PID controller with the parameters $K = 5$, $T_i = 10$ and $T_d = 2$.

- a. Draw a block diagram of the entire closed loop system including the disturbances. Determine the transfer functions from $R(s)$ to $Y(s)$, $D_1(s)$ to $Y(s)$ and $D_2(s)$ to $Y(s)$. What is the stationary error if there is no disturbance? (2 p)
- b. The increasing strain on the factory gives rise to ramp disturbances of $d_1(t) = \beta_1 t$, $d_2(t) = \beta_2 t$. What is the stationary error given these disturbances? (1.5 p)
- c. Save Christmas by suggesting an improvement to Santas naive PID controller, such that the factories now manages to produce the right amount of presents. (0.5 p)

Solution

- a. Transfer functions with given parameters

$$G_p(s) = \frac{1}{s^2 + s} \quad (23)$$

$$G_r(s) = \frac{20s^2 + 10s + 1}{2s} \quad (24)$$

$$G_o(s) = \frac{20s^2 + 10s + 1}{2s^3 + 2s^2} \quad (25)$$

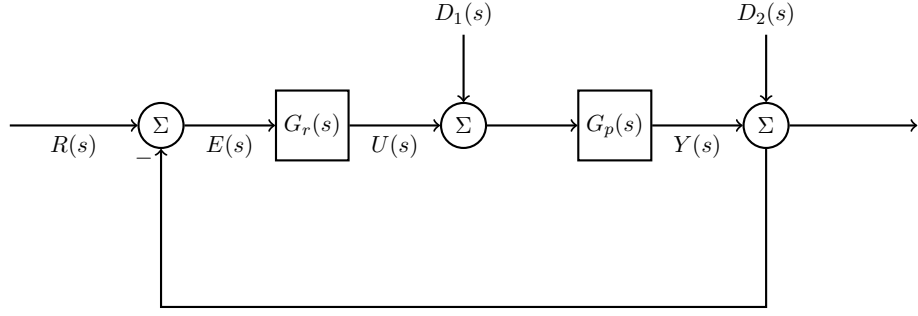


Figure 4: Control system with disturbances for Problem 5.

Transfer functions from R, D_1, D_2 to Y are given by

$$Y(s) = \frac{G_r G_p}{1 + G_r G_p} R(s) = \frac{20s^2 + 10s + 1}{2s^3 + 22s^2 + 10s + 1} \quad (26)$$

$$Y(s) = \frac{G_p}{1 + G_r G_p} D_1(s) = \frac{2s}{2s^3 + 22s^2 + 10s + 1} \quad (27)$$

$$Y(s) = \frac{1}{1 + G_r G_p} D_2(s) = \frac{2s^3 + 2s^2}{2s^3 + 22s^2 + 10s + 1} \quad (28)$$

Given no disturbances, the error can be written as

$$E(s) = R(s) - Y(s) = \frac{1}{1 + G_r G_p} R(s) = \frac{2s^3 + 2s^2}{2s^3 + 22s^2 + 10s + 1} R(s) \quad (29)$$

using the criterion for stability, the poles are positive if $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$, and we have a positive coefficient in front of s^3 , which is the case.

The final value theorem can thus be used.

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{2s^3 + 2s^2}{2s^3 + 22s^2 + 10s + 1} \frac{\alpha}{s^2} = 0 \quad (30)$$

b. The error given the ramp disturbances can be written as

$$E(s) = R(s) - Y(s) = \frac{1}{1 + G_r G_p} R(s) - \frac{G_p}{1 + G_r G_p} D_1(s) - \frac{1}{1 + G_r G_p} D_2(s) \quad (31)$$

$$= E_1(s) + E_2(s) + E_3(s) \quad (32)$$

Same stability condition holds as before, use final value theorem.

$$\lim_{s \rightarrow 0} sE_1(s) = \lim_{s \rightarrow 0} s \frac{2s^3 + 2s^2}{2s^3 + 22s^2 + 10s + 1} \frac{\alpha}{s^2} = 0 \quad (33)$$

$$\lim_{s \rightarrow 0} sE_2(s) = \lim_{s \rightarrow 0} s \frac{-2s}{2s^3 + 22s^2 + 10s + 1} \frac{\beta_1}{s^2} = -2\beta_1 \quad (34)$$

$$\lim_{s \rightarrow 0} sE_3(s) = \lim_{s \rightarrow 0} s \frac{-(2s^3 + 2s^2)}{2s^3 + 22s^2 + 10s + 1} \frac{\beta_2}{s^2} = -0 \quad (35)$$

$$(36)$$

Final error thus becomes

$$\lim_{t \rightarrow \infty} e(t) = -2\beta_1 \quad (37)$$

- c. The error occurs as the integrator in the plant does not affect the disturbance on the control signal. To remedy this the controller could be fitted with a second integrator.

6. Consider the following four transfer functions.

(1) $G_1(s) = (s^2 + 2s + 1)^{-1}e^{-2s}$

(2) $G_2(s) = -(s - 4)(s^2 + 4s + 4)^{-1}$

(3) $G_3(s) = (s + 3)(s^2 + s + 2)^{-1}$

(4) $G_4(s) = (s^2 + 4)^{-1}$

- a. Pair up the transfer functions (1)-(4), with a Bode diagram (A)-(D), and a Nyquist diagram (i)-(iv) in Figure 5 and Figure 7. Each transfer function and curve in the Bode/Nyquist diagrams should be used exactly once. Don't forget to motivate your answer! (2 p)

- b. Sketch the step-responses of each transfer function (1)-(4), and write a short comment motivating each sketched step response. (2 p)

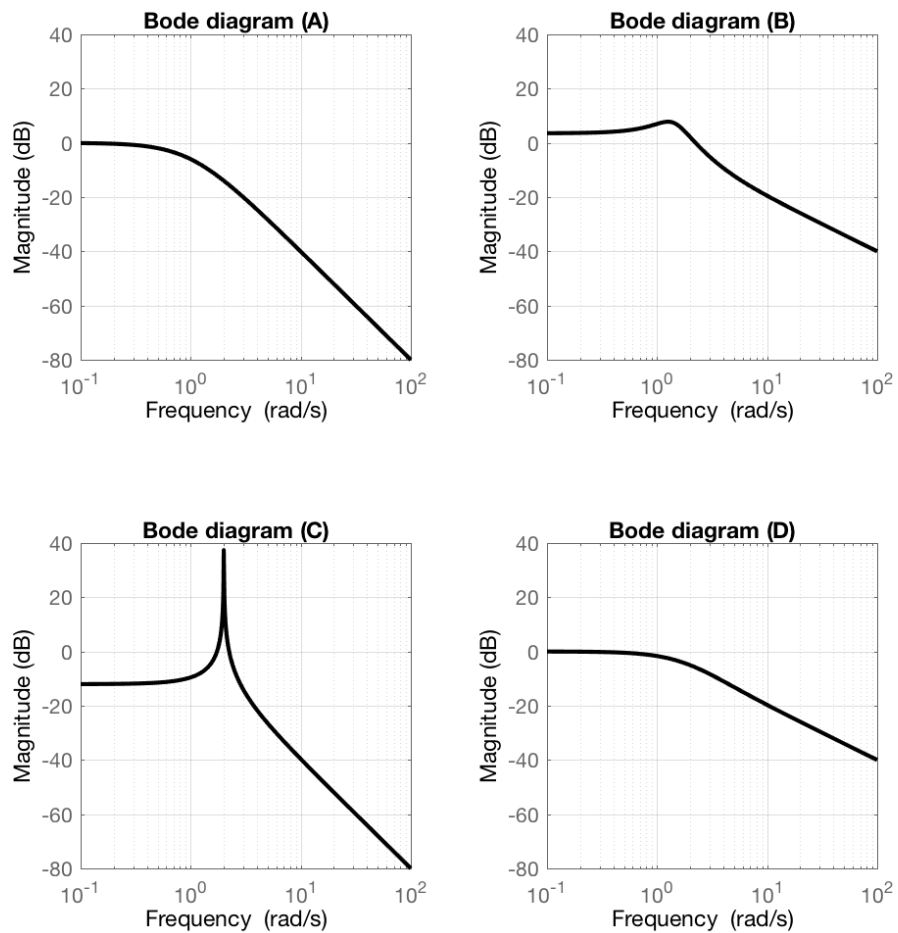
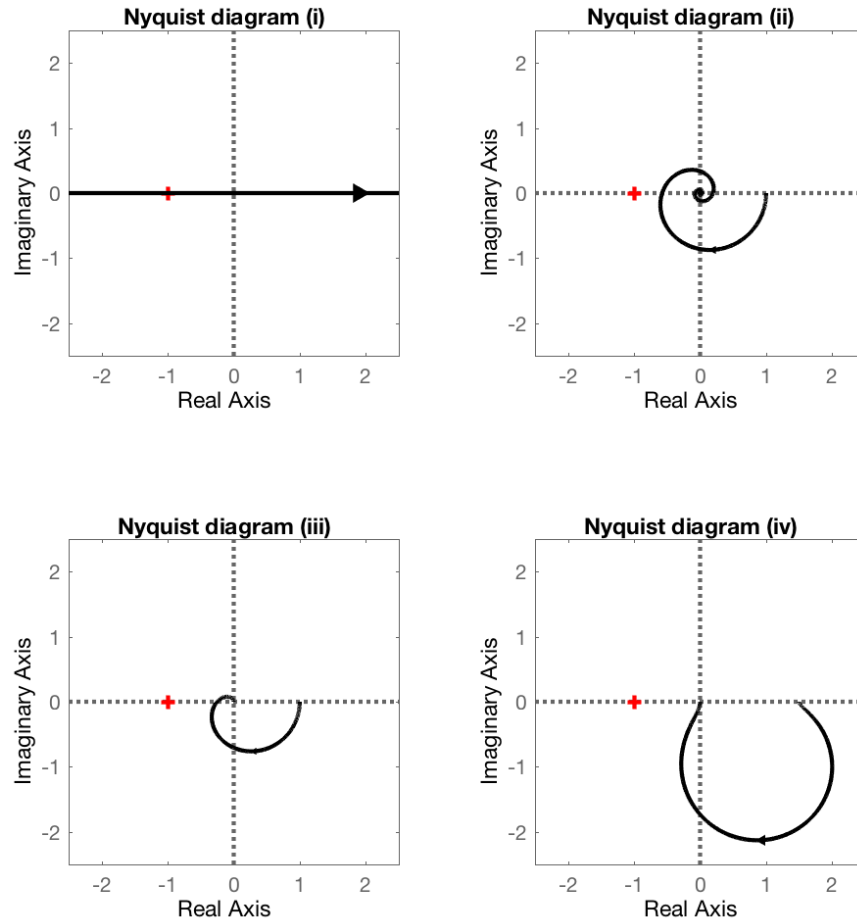


Figure 5: Bode diagrams for Problem 6.



Figur 6: Nyquist diagrams for Problem 6.

Solution

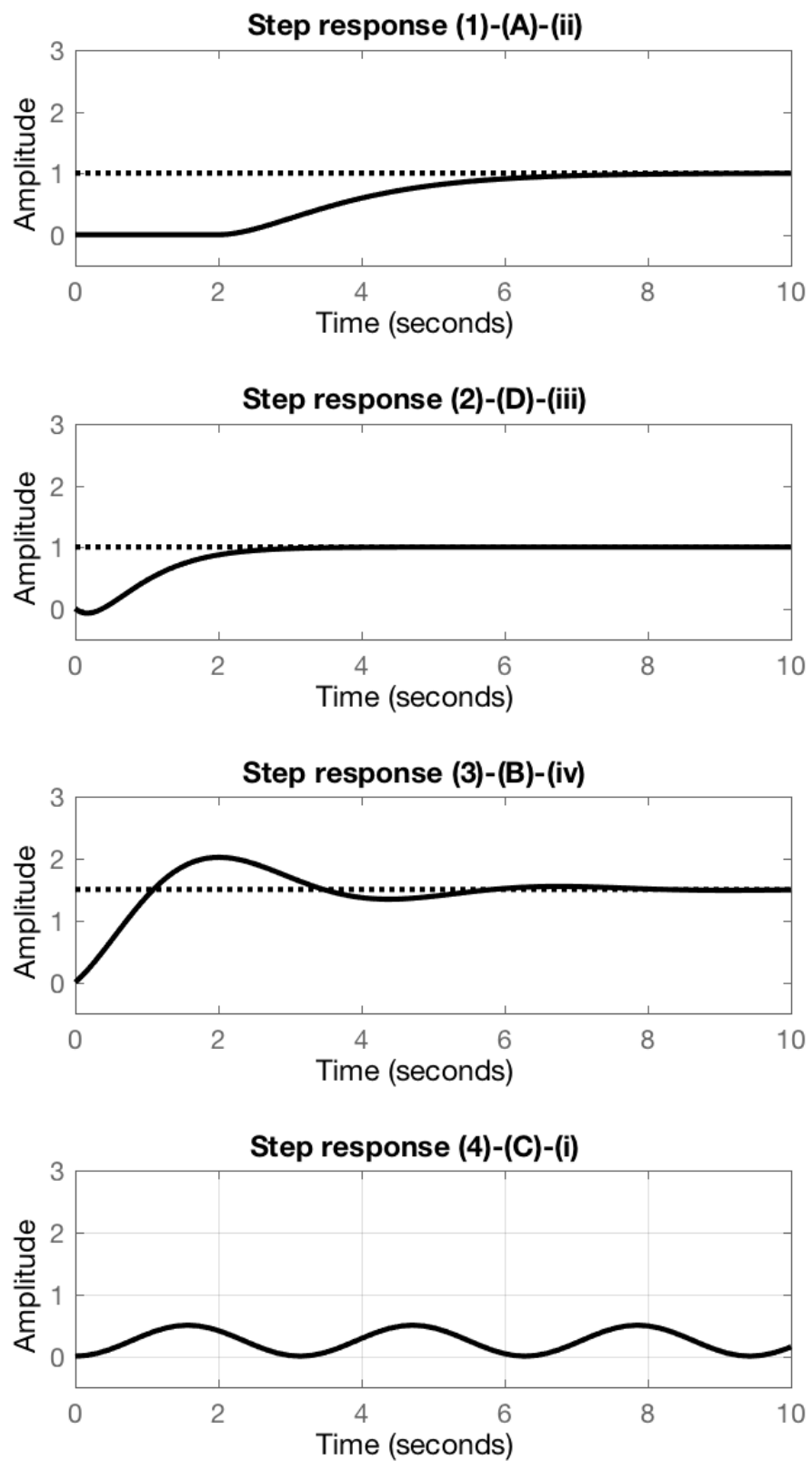
- a. (1) The system $G_1(s)$ has a time-delay, which is invisible in the magnitude diagram. However, we know that the static gain of this transfer function is $G_1(0) = 1$, *i.e.*, a magnitude of 0 dB. Furthermore, we know that the system has a double pole and no zero, *i.e.* a pole excess of 2, meaning that it will have a constant amplification of low frequencies and a roll-off of slope 2 at higher frequencies. The only solution which fits the bill is the red curve (A). Furthermore, the only curve which encircles the origin multiple times is the blue curve, *i.e.*, the Nyquist plot (ii). (0.5 p)
- (2) The system $G_2(s)$ has a pole excess of 1, and we would expect a rolloff of slope 1 at higher frequencies. Furthermore, we note that the system is critically damped, with a double pole in -2 , so it should not have a resonance peak. In addition, it has a static gain of 1. Consequently, the system corresponds to the magnitude plot (D) in the bode diagram, and the Nyquist plot (iii). (0.5 p)

- (3) The system $G_3(s)$ has a pole excess of 1, and we would expect a rolloff of slope 1 at higher frequencies. Furthermore, we note that the system is under-damped, with poles in $-0.5 \pm i\sqrt{1.75}$, so it should not have a slight resonance peak at $\omega = \sqrt{2}$. In addition, it has a static gain of 1.5, which is roughly 3.5 dB. Clearly, this corresponds to the magnitude plot (B), and the corresponding Nyquist diagram is found in (iv). (0.5 p)
- (4) The system $G_4(s)$ has a pole excess of 2, and we would expect a rolloff of slope 2 at higher frequencies. Furthermore, we note that the system is only marginally stable, with two imaginary poles in $\pm 2i$, so it should not have a significant resonance peak at $\omega = 2$. Clearly, this corresponds to the magnitude plot (C). Since the system poles are purely imaginary, we have that $G(i\omega) = (4 - \omega^2)^{-1} \in \mathbb{R}$ for all frequencies ω , thus the Nyquist curve should reside on the real axis for all ω , corresponding to (i). (0.5 p)

In summary

- (1)-(A)-(ii)
 - (2)-(D)-(iii)
 - (3)-(B)-(iv)
 - (4)-(C)-(i)
- b. (1) To summarise the previous points, we have a static gain of 1, a time-delay of 2, critical damping (no overshoot) with a relatively slow rise-time (absolute value of poles is 1). Knowing this, we can draw something similar to the red curve. (0.5 p)
- (2) Now we have a static gain of 1, critical damping (no overshoot) with a slightly faster rise-time (absolute value of poles is 2). In addition, we have a zero in the right half plane, and our step like to the blue curve. (0.5 p)
- (3) Now we have a static gain of 1.5, the system is quite under-damped (significant overshoot) with a moderately fast rise-time (absolute value of poles is $\sqrt{2}$). Consequently, the system is faster than G_1 , and slower than G_2 . In addition, we have a zero in the left half plane. With this information, we can sketch something like the black curve. (0.5 p)
- (4) This system is only marginally stable, and we expect a sinusoidal behaviour in the system response with a period of 2 [rad/s], i.e., a period of $T = 2\pi/2 = \pi$. Any sinusoidal behaviour with roughly this period will score a point. (0.5 p)

Good luck!



Figur 7: Step responses and solution to Problem 6.