Step Response Analysis. Frequency Response, Relation Between Model Descriptions

Automatic Control, Basic Course, Lecture 3

November 8, 2018

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Content

Step Response Analysis

Step Response

From the last lecture, we know that if the input $u(t)$ is a **step**, then the output in the Laplace domain is

$$Y(s) = G(s)U(s) = G(s)\frac{\mathbf{1}}{\mathbf{s}}$$

It is possible to do an inverse transform of $Y(s)$ to get $y(t)$, but is it possible to claim things about $y(t)$ by only studying $Y(s)$?

We will study **how the poles affects the step response**. (The zeros will be discussed later).

Initial and Final Value Theorem

Let $F(s)$ be the Laplace transformation of $f(t)$, i.e., $F(s) = \mathcal{L}(f(t))(s)$. **Given that the limits below exist**[1], it holds that:

Initial value theorem	$\lim_{t\to 0} f(t) = \lim_{s\to +\infty} sF(s)$
Final value theorem	$\lim_{t\to +\infty} f(t) = \lim_{s\to 0} sF(s)$

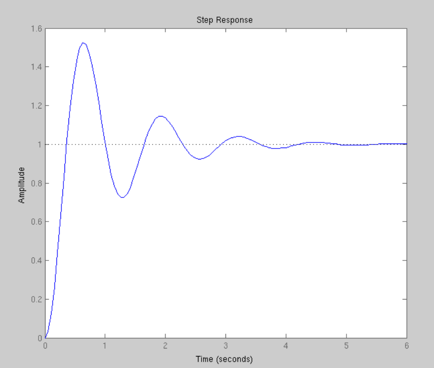
For a step response we have that:

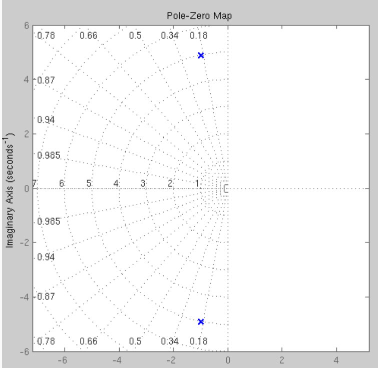
$$\lim_{t\to +\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0)$$

[1] Q: When can we NOT apply the Final value theorem?

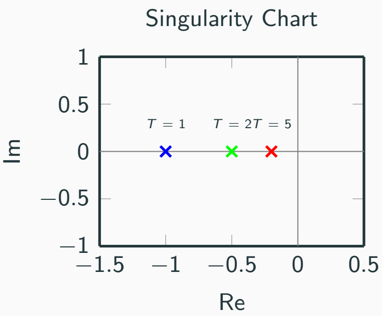
Some useful matlab commands

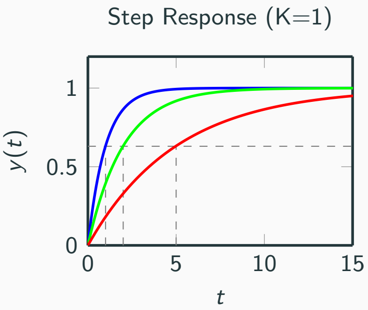
```
>> s=tf('s'); % enables to use s as transfer fcn
>> z=0.2; w0=5;
>> G= w0^2 / (s^2 + 2*z*w0*s + w0^2 )
>> step(G)
>>
>> pzmap(G) % pole-zero map
```





First Order System





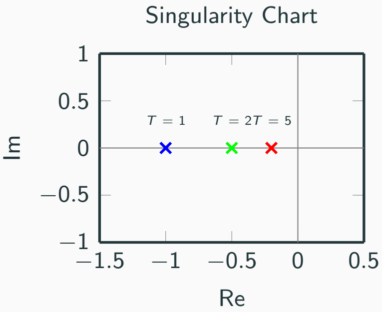
$$G(s) = \frac{K}{1 + sT}$$

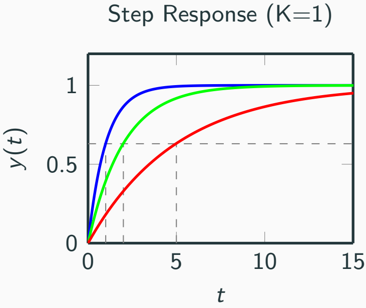
One pole in $s = -1/T$

Step response:

$$Y(s) = G(s)\frac{1}{s} = \frac{K}{s(1+sT)} \quad \xrightarrow{\mathcal{L}^{-1}} \quad y(t) = K\left(1 - e^{-t/T}\right), \ \mathbf{t \geq 0}$$

First Order System



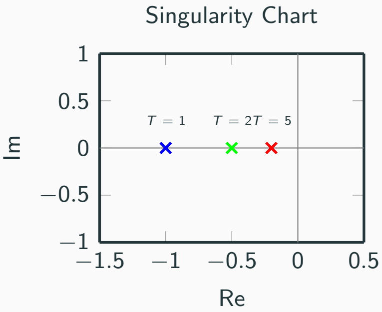


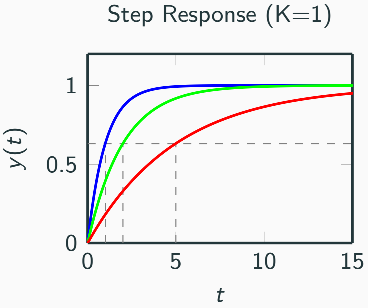
$$G(s) = \frac{K}{1 + sT}$$

Final value:

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \cdot \frac{K}{s(1 + sT)} = K$$

First Order System





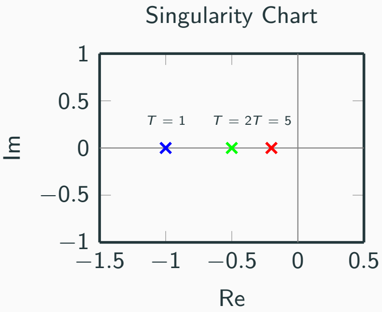
$$G(s) = \frac{K}{1 + sT}$$

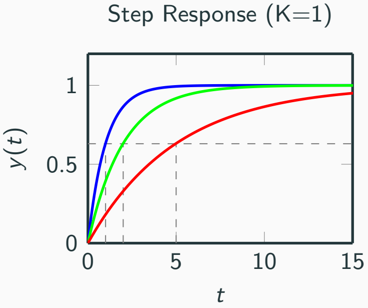
T is called the time-constant:

$$y(T) = K(1 - e^{-T/T}) = K(1 - e^{-1}) \approx 0.63K$$

i.e., T is the time it takes for the step response to reach 63% of its final value

First Order System



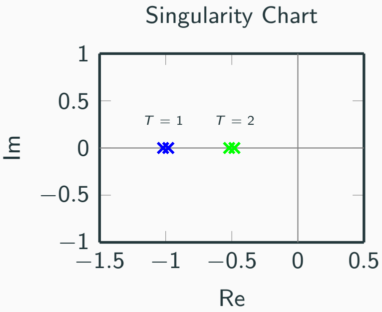


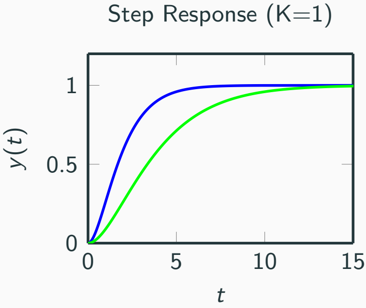
$$G(s) = \frac{K}{1 + sT}$$

Derivative at zero:

$$\lim_{t\to 0} \dot{y}(t) = \lim_{s\to +\infty} s \cdot sY(s) = \lim_{s\to +\infty} \frac{s^2 K}{s(1+sT)} = \frac{K}{T}$$

Second Order System With Real Poles



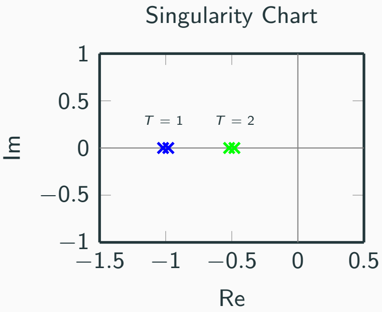


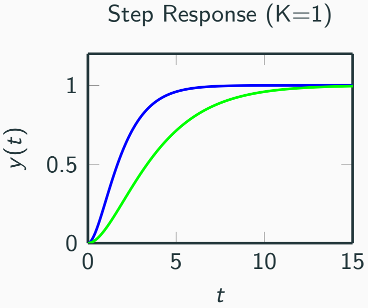
$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Poles in $s = -1/T_1$ and $s = -1/T_2$. Step response:

$$y(t) = \begin{cases} K\left(1 - \frac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2}\right), \ \mathbf{t \geq 0} & T_1 \neq T_2 \\ K\left(1 - e^{-t/T} - \frac{t}{T} e^{-t/T}\right), \ \mathbf{t \geq 0} & T_1 = T_2 = T \end{cases}$$

Second Order System With Real Poles



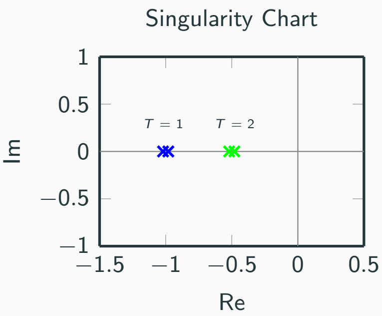


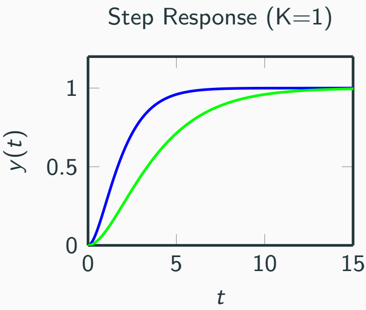
$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Final value:

$$\lim_{t \to +\infty} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{sK}{s(1 + sT_1)(1 + sT_2)} = K$$

Second Order System With Real Poles





$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Derivative at zero:

$$\lim_{t \to 0} \dot{y}(t) = \lim_{s \to +\infty} s \cdot sY(s) = \lim_{s \to +\infty} \frac{s^2 K}{s(1 + sT_1)(1 + sT_2)} = 0$$

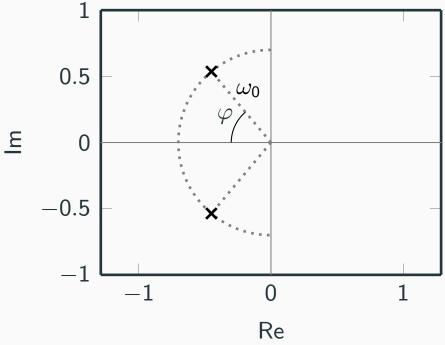
Second Order System With Complex Poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad 0 < \zeta < 1$$

Relative damping ζ, related to the angle φ

$$\zeta = \cos(\varphi)$$

Singularity Chart



Second Order System With Complex Poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad 0 < \zeta < 1$$

Inverse transformation for step response yields:

$$\begin{aligned} y(t) &= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin\left(\omega_0\sqrt{1-\zeta^2}t + \arccos\zeta\right)\right) \\ &= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin\left(\omega_0\sqrt{1-\zeta^2}t + \arcsin(\sqrt{1-\zeta^2})\right)\right), \mathbf{t \geq 0} \end{aligned}$$

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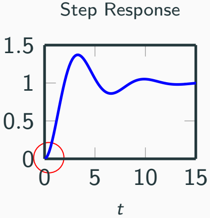
Exercise: Check of correct starting point of step response.

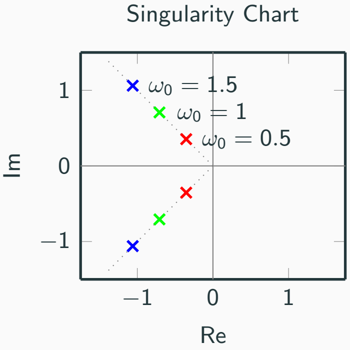
$$\begin{aligned} y(0) &= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{0} \sin\left(\omega_0\sqrt{1-\zeta^2}0 + \arcsin(\sqrt{1-\zeta^2})\right)\right) \\ &= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} \cdot \sqrt{1-\zeta^2}\right) \\ &= 0 \end{aligned}$$

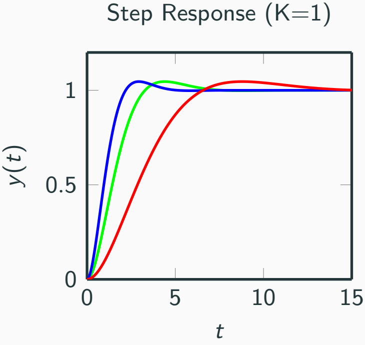


Second Order System With Complex Poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad 0 < \zeta < 1$$

Changing fq ω_0

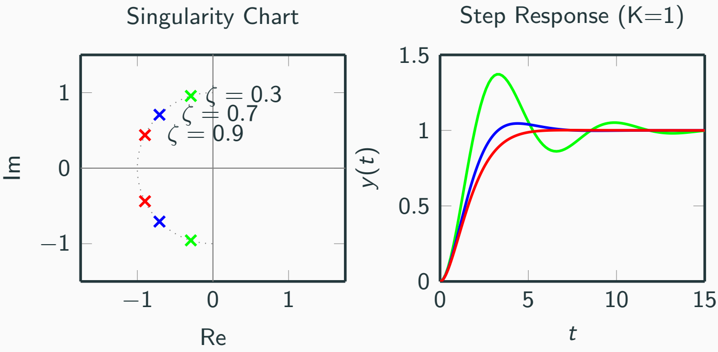




Second Order System With Complex Poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad 0 < \zeta < 1$$

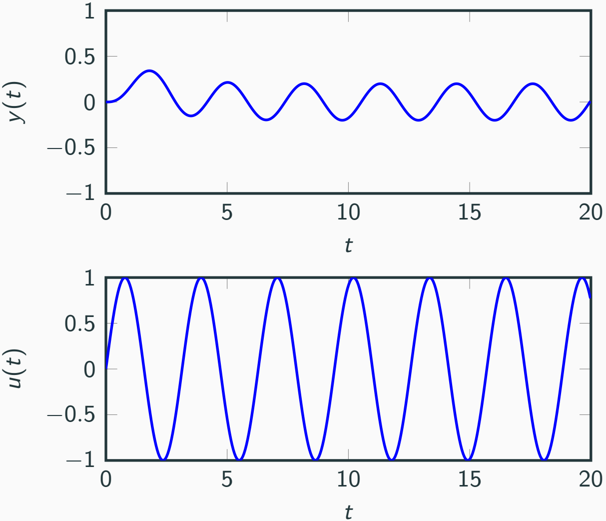
Changing damping ζ



Frequency Response

Sinusoidal Input

Given a transfer function $G(s)$, what happens if we let the input be $u(t) = \sin(\omega t)$?



Sinusoidal Input

It can be shown that if the input is $u(t) = \sin(\omega t)$, the output[2] will be

$$y(t) = A \sin(\omega t + \varphi)$$

where

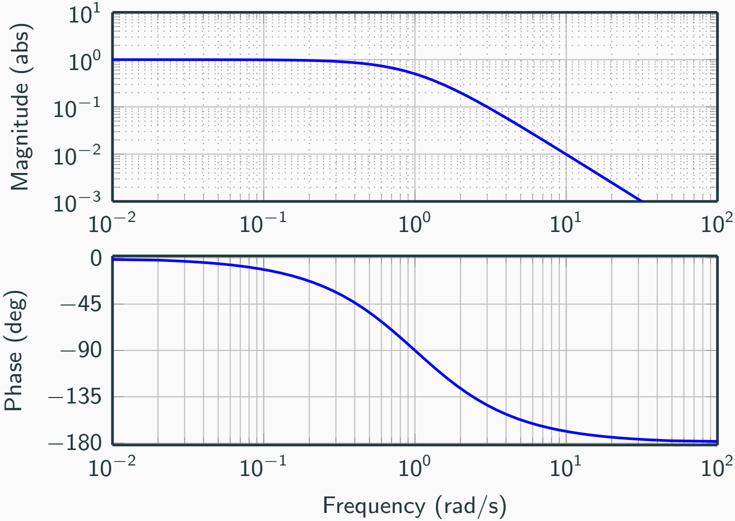
$$A = |G(i\omega)|$$
$$\varphi = \arg G(i\omega)$$

So if we determine a and φ for different frequencies ω, we have a description of the transfer function.

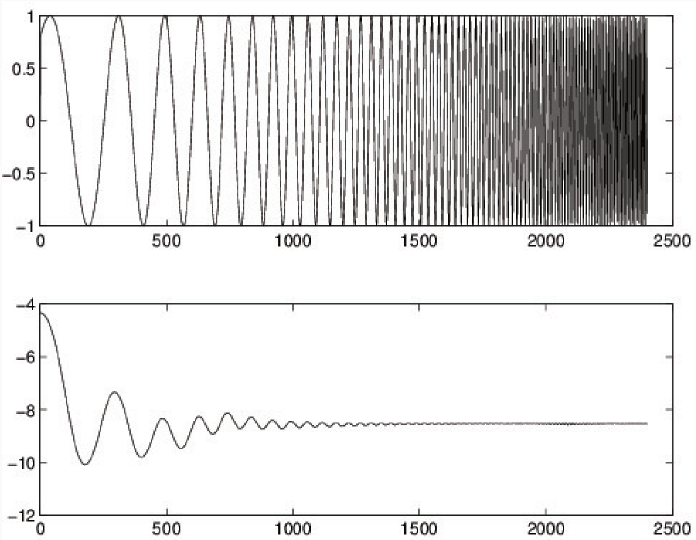
[2]after the transient has decayed

Bode Plot

Idea: Plot $|G(i\omega)|$ and $\arg G(i\omega)$ for different frequencies ω.

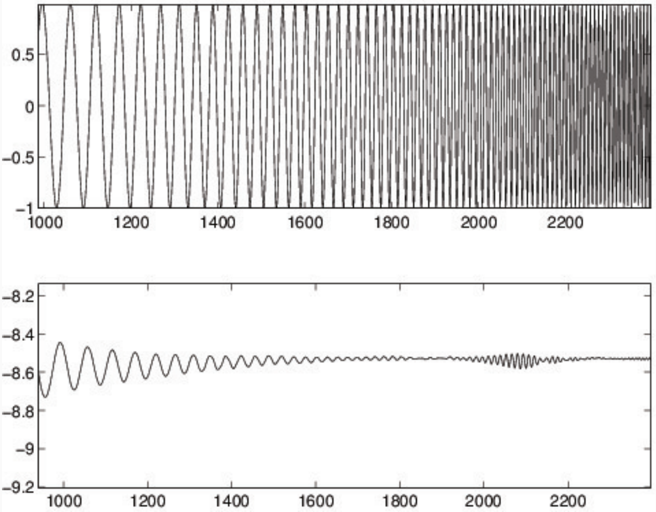


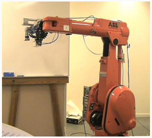
Sinusoidal Input-Output: example with frequency sweep (chirp)



Resonance frequency of industrial robot IRB2000 visible in data.

Sinusoidal Input-Output: example with frequency sweep (chirp)





Resonance frequency of industrial robot IRB2000 visible in data.

Bode Plot - Products of Transfer Functions

Let

$$G(s) = G_1(s)G_2(s)G_3(s)$$

then

$$\begin{aligned} \log|G(i\omega)| &= \log|G_1(i\omega)| + \log|G_2(i\omega)| + \log|G_3(i\omega)| \\ \arg G(i\omega) &= \arg G_1(i\omega) + \arg G_2(i\omega) + \arg G_3(i\omega) \end{aligned}$$

This means that we can construct Bode plots of transfer functions from simple "building blocks" for which we know the Bode plots.

Bode Plot of $G(s) = K$

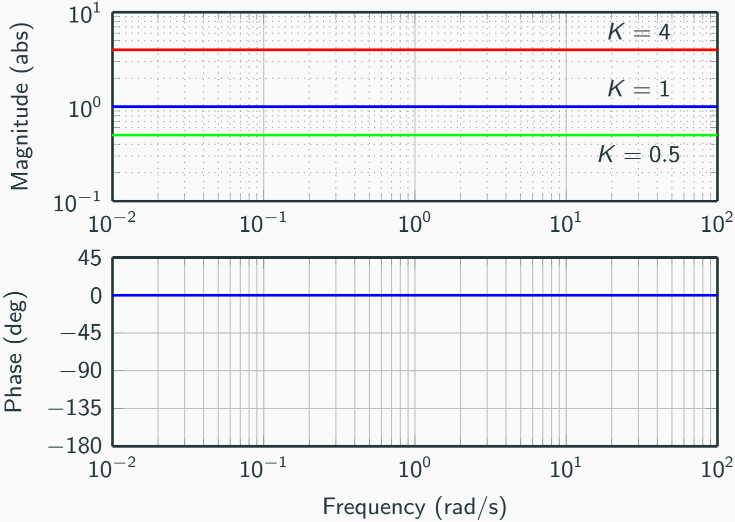
If

$$G(s) = K$$

then

$$\log|G(i\omega)| = \log(|K|)$$
$$\arg G(i\omega) = 0 \qquad (\textit{if } K > 0, \textit{ else } +180 \textit{ or } -180 \deg)$$

Bode Plot of $G(s) = K$



Bode Plot of $G(s) = s^n$

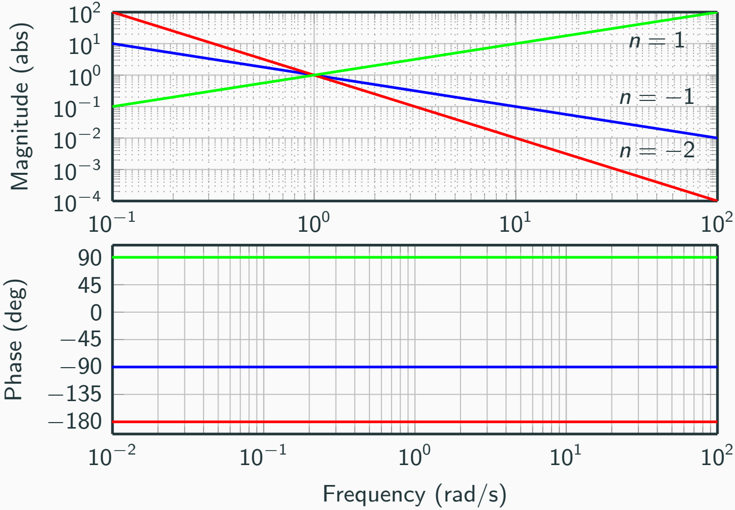
If

$$G(s) = s^n$$

then

$$\log|G(i\omega)| = n\log(\omega)$$
$$\arg G(i\omega) = n\frac{\pi}{2}$$

Bode Plot of $G(s) = s^n$



Bode Plot of $G(s) = (1 + sT)^n$

If

$$G(s) = (1 + sT)^n$$

then

$$\log|G(i\omega)| = n\log(\sqrt{1+\omega^2T^2})$$
$$\arg G(i\omega) = n\arg(1 + i\omega T) = n\arctan(\omega T)$$

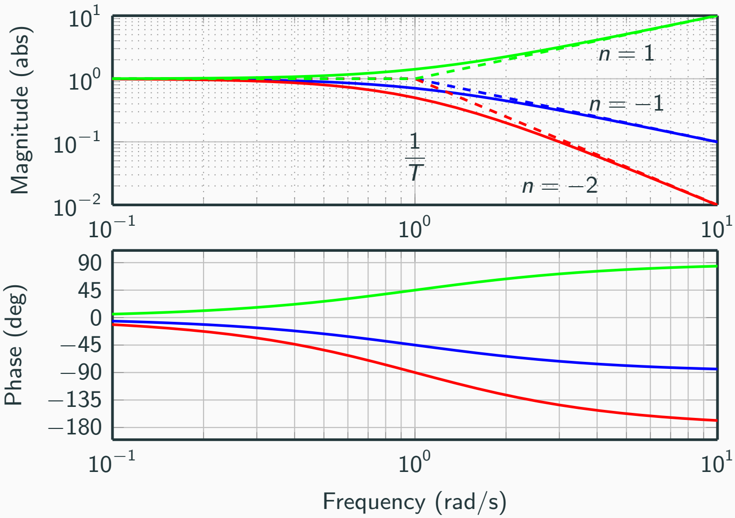
For small ω

$$\log|G(i\omega)| \to 0$$
$$\arg G(i\omega) \to 0$$

For large ω

$$\log|G(i\omega)| \to n\log(\omega T)$$
$$\arg G(i\omega) \to n\frac{\pi}{2}$$

Bode Plot of $G(s) = (1 + sT)^n$



Bode Plot of $G(s) = (1 + 2\zeta s/\omega_0 + (s/\omega_0)^2)^n$

$$G(s) = (1 + 2\zeta s/\omega_0 + (s/\omega_0)^2)^n$$

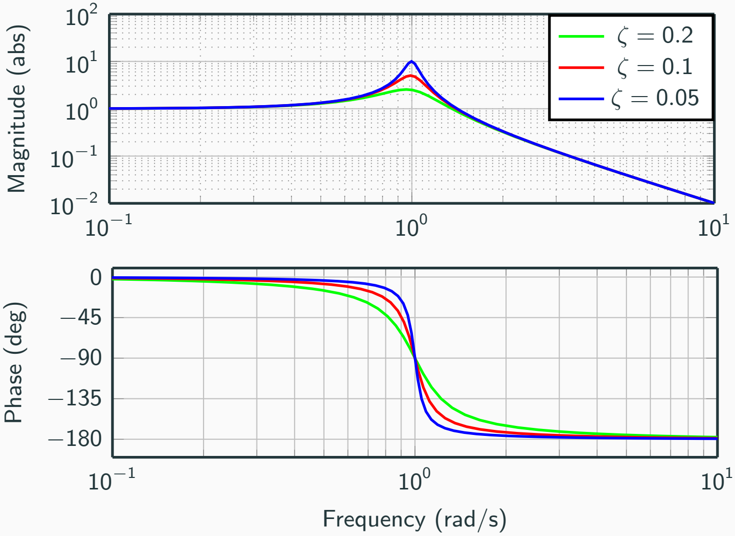
For small ω

$$\log|G(i\omega)| \to 0$$
$$\arg(i\omega) \to 0$$

For large ω

$$\log|G(i\omega)| \to 2n\log\left(\frac{\omega}{\omega_0}\right)$$
$$\arg G(i\omega) \to n\pi$$

Bode Plot of $G(s) = (1 + 2\zeta s/\omega_0 + (s/\omega_0)^2)^n$



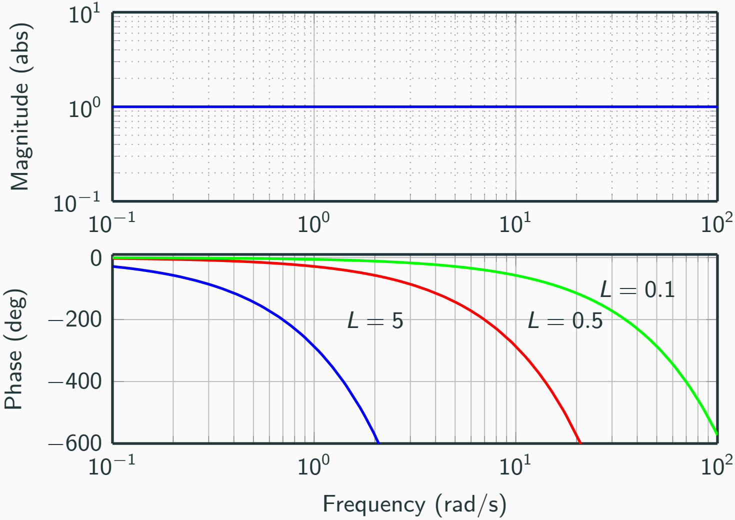
Bode Plot of $G(s) = e^{-sL}$

$$G(s) = e^{-sL}$$

Describes a pure time delay with delay L, i.e, $y(t) = u(t - L)$

$$\log|G(i\omega)| = 0$$
$$\arg G(i\omega) = -\omega L$$

Bode Plot of $G(s) = e^{-sL}$

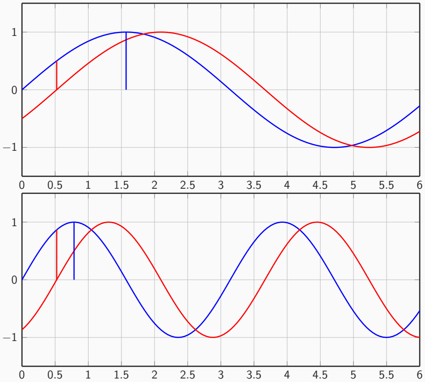


Bode Plot of $G(s) = e^{-sL}$

Same delay may appear as **different phase lag** for different frequencies!

Example

Delay ≈ 0.52 sec between input and output.



(Upper): Period time $= 2\pi \approx$ 6.28 sec. Delay represents phase lag of $\frac{0.52}{6.28} \cdot 360 \approx 30\,\text{deg}$

(Lower): Period time $= \pi \approx$ 3.14 sec. Delay represents phase lag of $\frac{0.5}{3.14} \cdot 360 \approx 60\,\text{deg}$.

Bode Plot of $G(s) = e^{-sL}$

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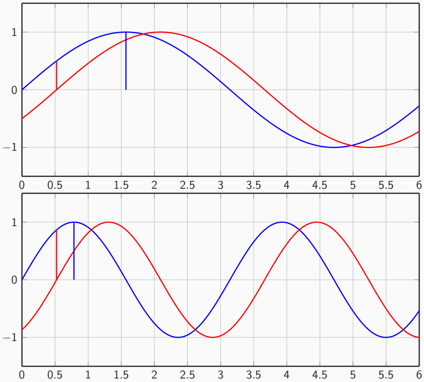


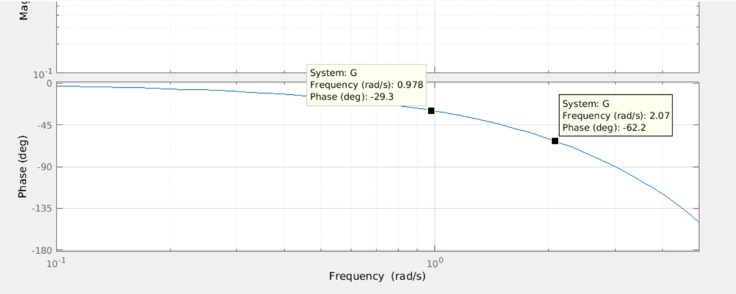
(Upper): Period time $= 2\pi \approx 6.28$ sec. Delay represents phase lag of $\frac{0.52}{6.28} \cdot 360 \approx 30\,\text{deg}$

(Lower): Period time $= \pi \approx 3.14$ sec. Delay represents phase lag of $\frac{0.5}{3.14} \cdot 360 \approx 60\,\text{deg}$.

Bode Plot of $G(s) = e^{-sL}$

Check phase in Bode diagram for $e^{-0.52s}$ for

- $\sin(t) \Rightarrow \omega = 1.0$ rad/s
- $\sin(2t) \Rightarrow \omega = 2.0$ rad/s




```
>> s=tf('s')
>> G=exp(-0.52*s);
>> bode(G,0.1 ,5) % Bode plot in frequency-range [0.1 .. 5] rad/s
```

Bode Plot of Composite Transfer Function

Example

Draw the Bode plot of the transfer function

$$G(s) = \frac{100(s+2)}{s(s+20)^2}$$

First step, write it as product of simple transfer functions:

$$G(s) = \frac{100(s+2)}{s(s+20)^2} = 0.5 \cdot s^{-1} \cdot (1+0.5s) \cdot (1+0.05s)^{-2}$$

Then determine the corner frequencies (break points):

$$G(s) = \frac{100(s+2)}{s(s+20)^2} = 0.5 \cdot s^{-1} \cdot \overbrace{(1+0.5s)}^{w_{c_1}=2} \cdot \overbrace{(1+0.05s)^{-2}}^{w_{c_2}=20}$$

Bode Plot of Composite Transfer Function

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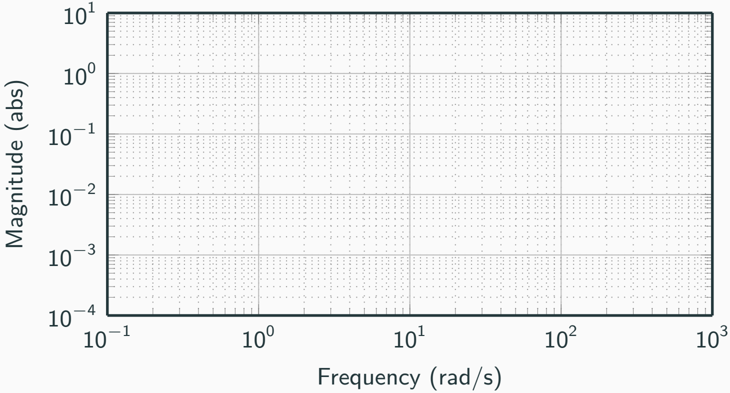
Sort from LOW to HIGH frequencies:

Start with LOW frequencies

(make sure the other TFs asymptotically reduce to **1**).

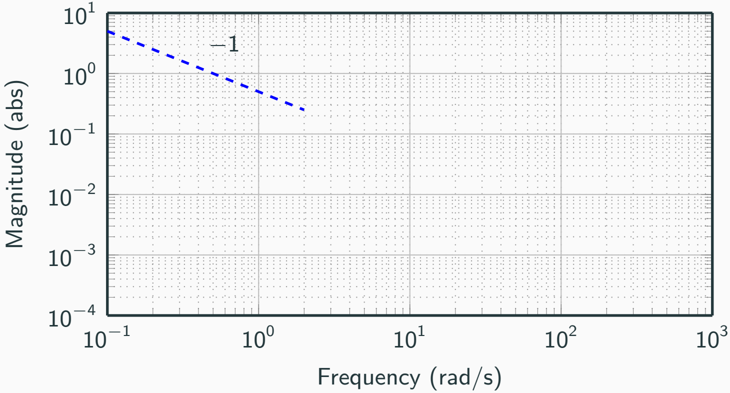
Bode Plot of Composite Transfer Function

$$G(s) = \frac{100(s+2)}{s(s+20)^2} = 0.5 \cdot s^{-1} \cdot \overbrace{(1+0.5s)}^{w_{c_1}=2} \cdot \overbrace{(1+0.05s)^{-2}}^{w_{c_2}=20}$$



Bode Plot of Composite Transfer Function

$$G(s) = \frac{100(s+2)}{s(s+20)^2} = 0.5 \cdot s^{-1} \cdot \overbrace{(1+0.5s)}^{w_{c_1}=2} \cdot \overbrace{(1+0.05s)^{-2}}^{w_{c_2}=20}$$



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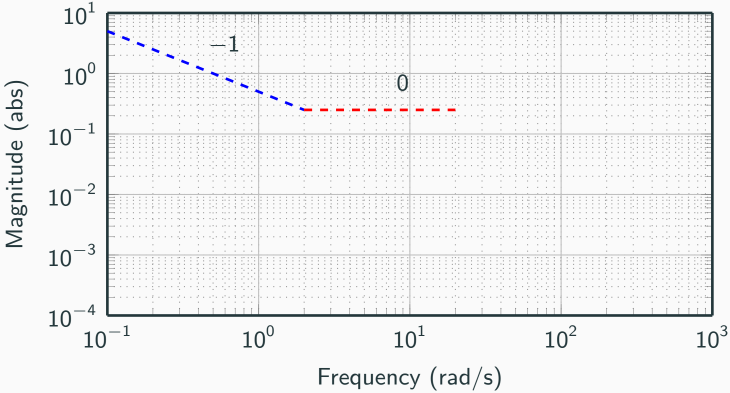
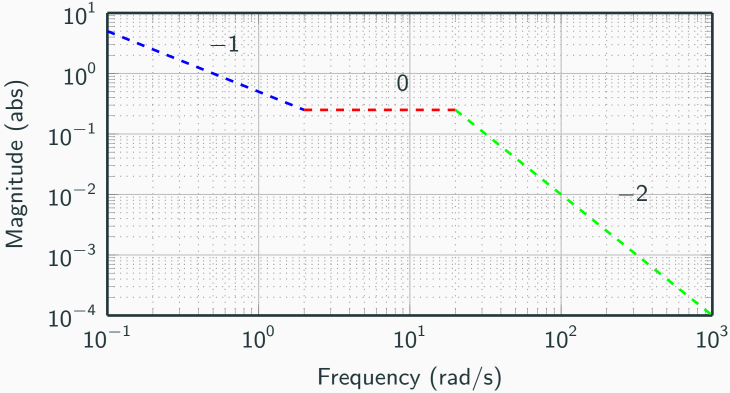


Bode Plot of Composite Transfer Function

$$G(s) = \frac{100(s+2)}{s(s+20)^2} = 0.5 \cdot s^{-1} \cdot \overbrace{(1+0.5s)}^{w_{c_1}=2} \cdot \overbrace{(1+0.05s)^{-2}}^{w_{c_2}=20}$$



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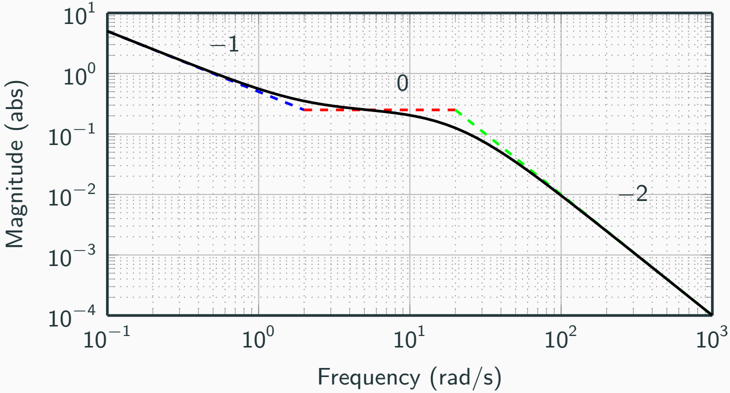

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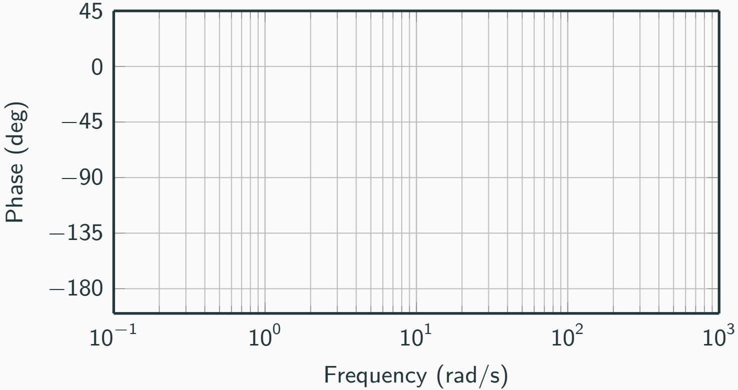

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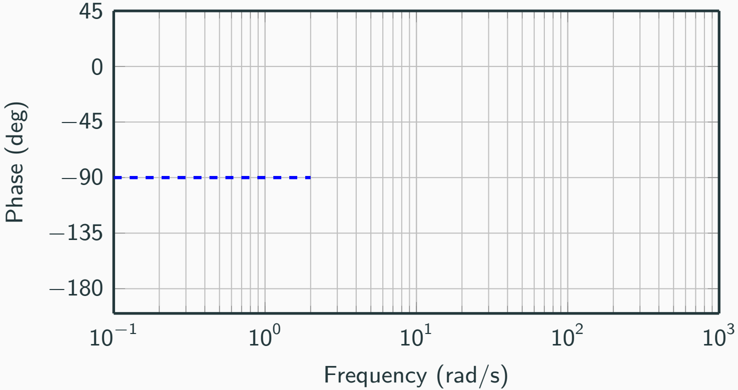

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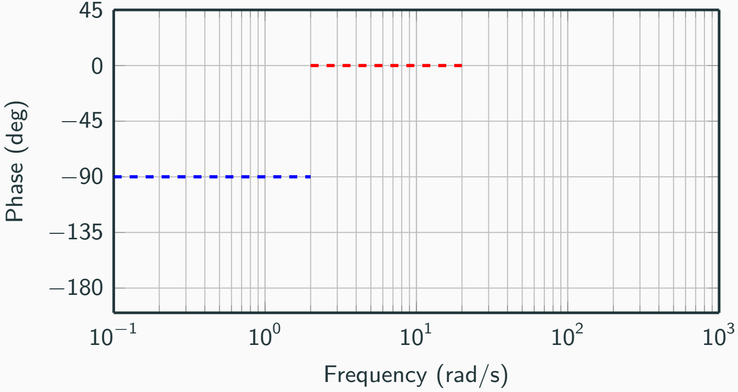

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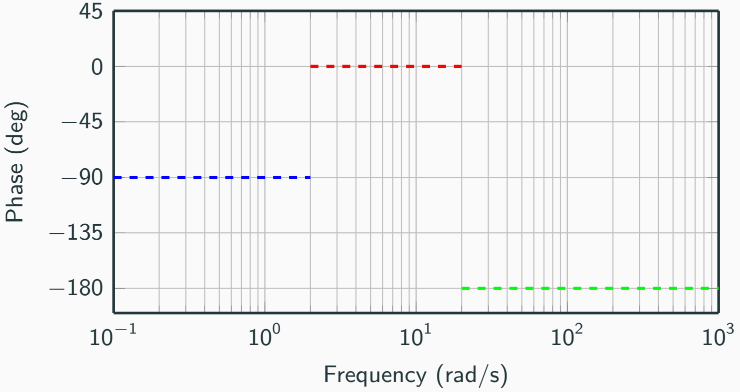

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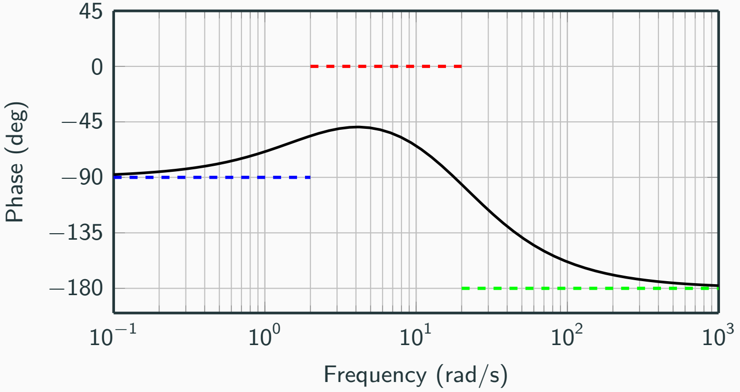
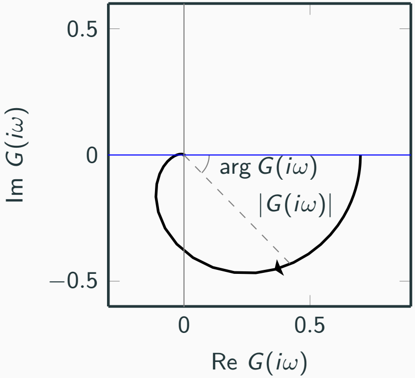


Nyquist Plot

By removing the frequency information, we can plot the transfer function in one plot instead of two.



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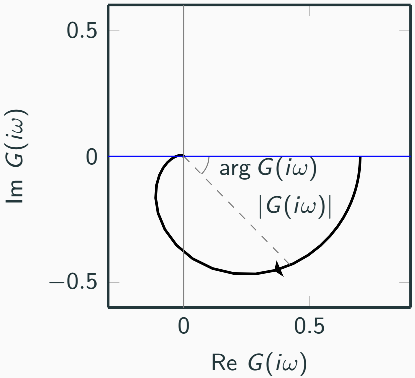
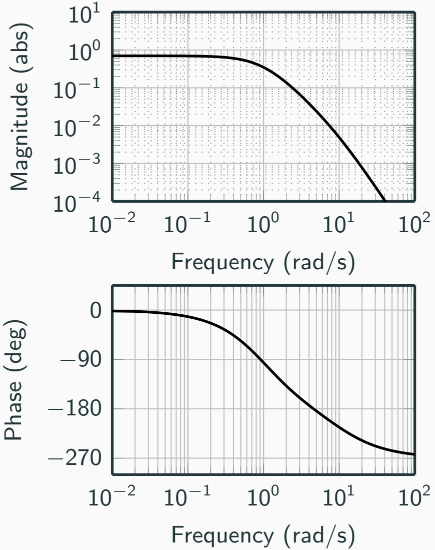


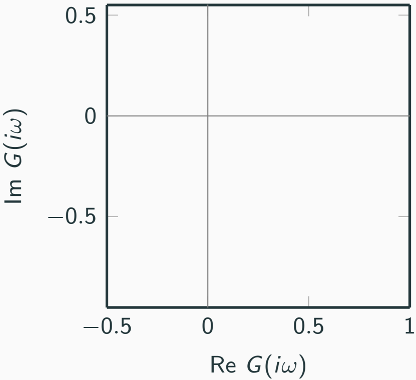
Split the transfer function into real and imaginary part:

$$G(s) = \frac{1}{1+s} \quad G(i\omega) = \frac{1}{1+i\omega} = \frac{1}{1+\omega^2} - i\frac{\omega}{1+\omega^2}$$

Is this the transfer function in the plot above?

From Bode Plot to Nyquist Plot





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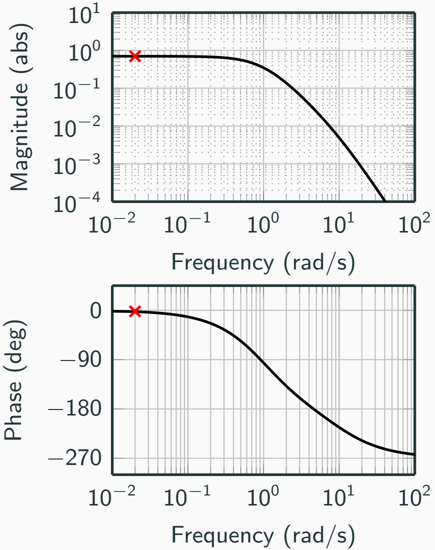

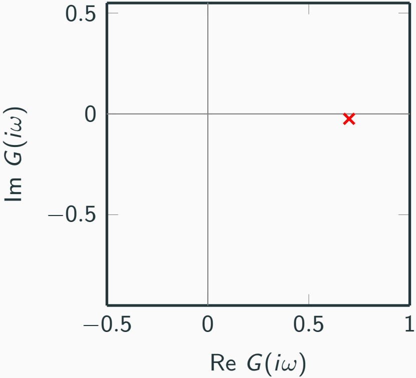

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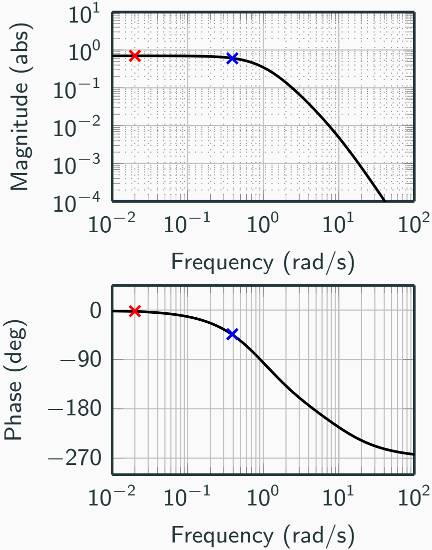

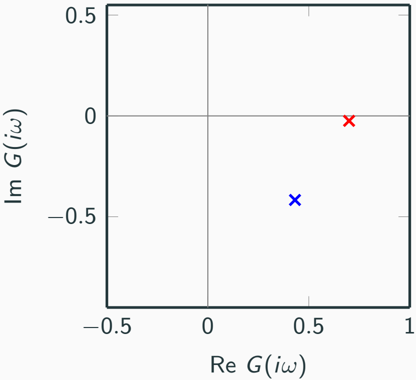

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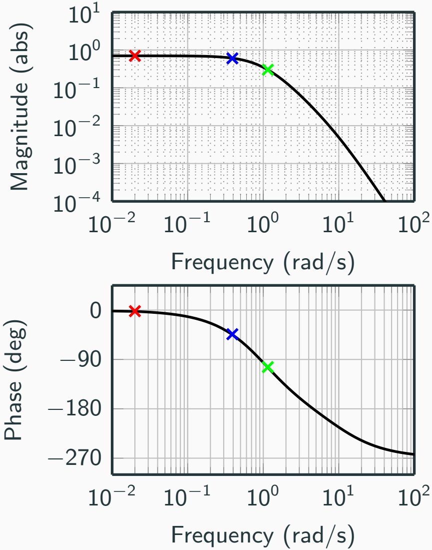

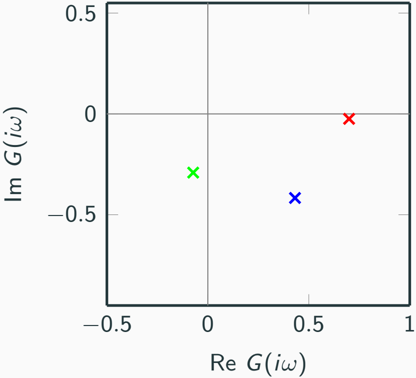

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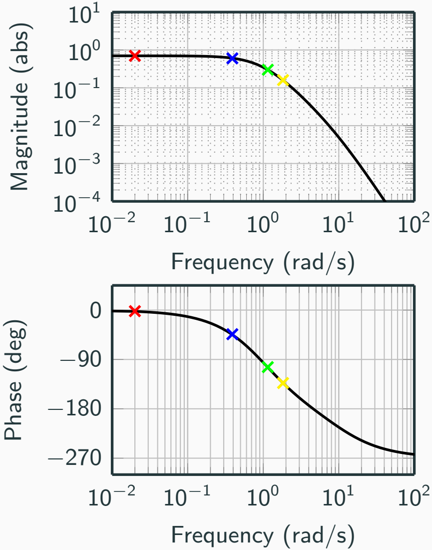
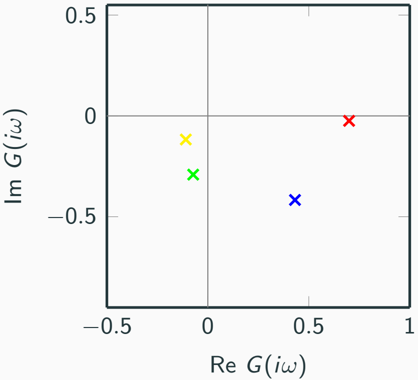
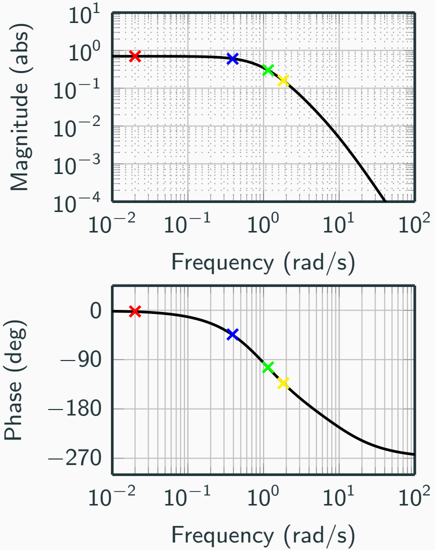
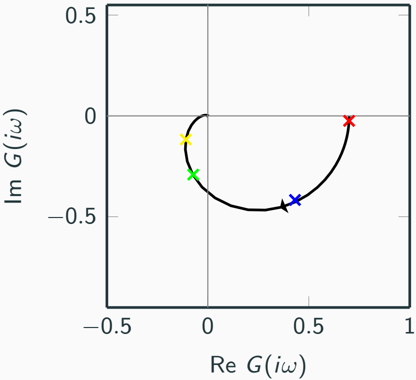




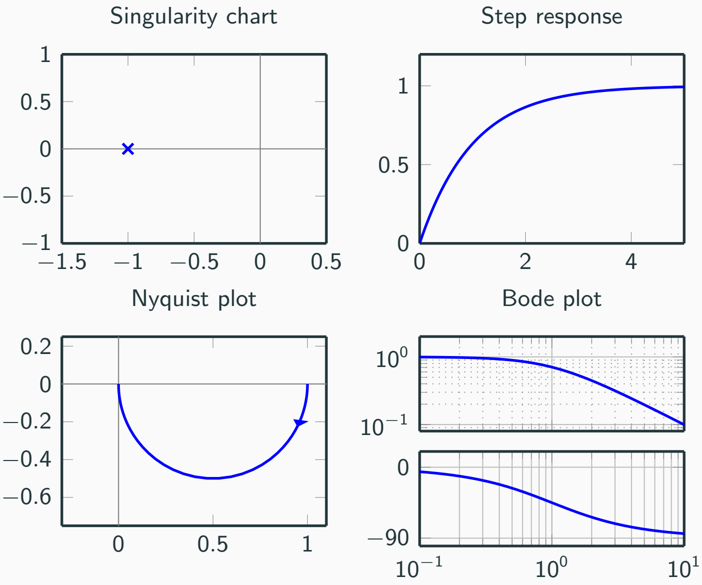
From Bode Plot to Nyquist Plot



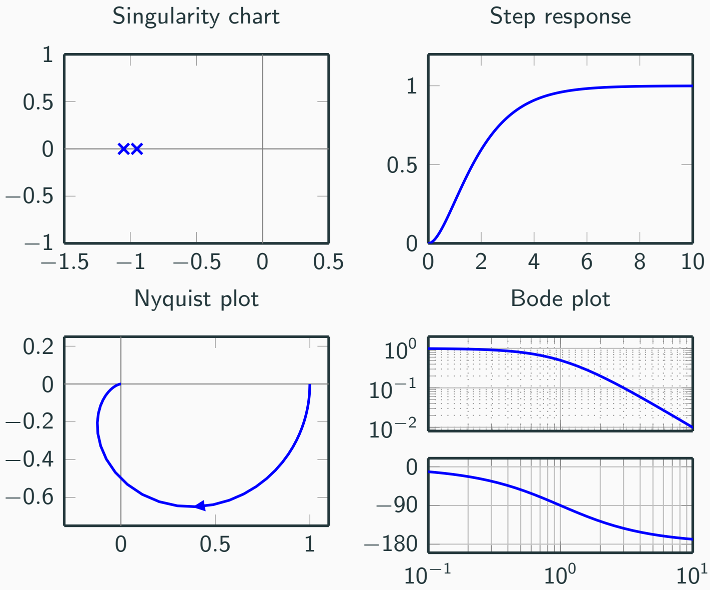


Relation between Model Descriptions

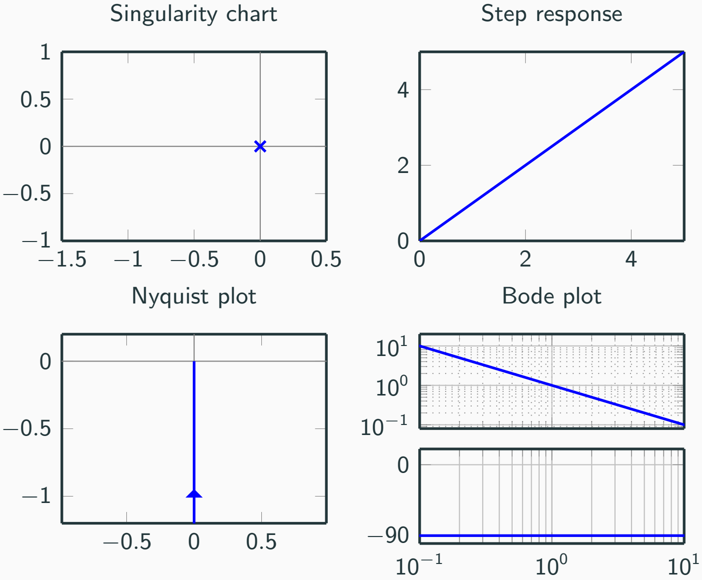
Single-capacitive Processes $\frac{K}{sT+1}$



Multi-capacitive Processes $\frac{K}{(sT_1+1)(sT_2+1)}$

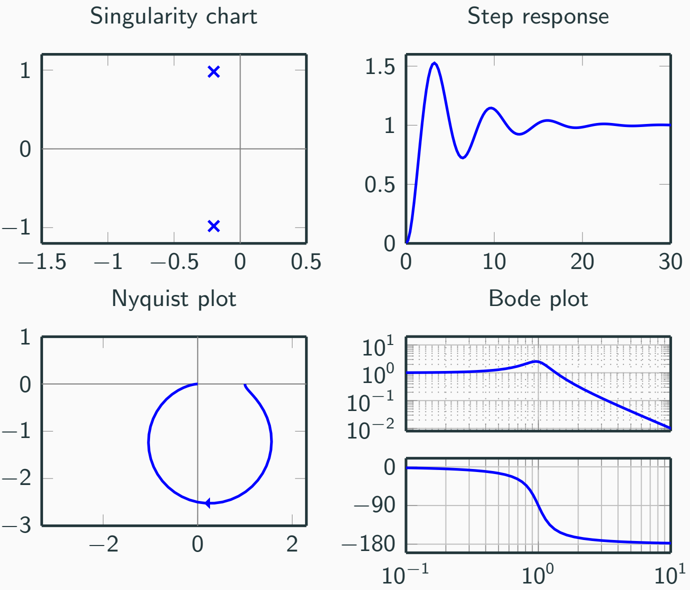


Integrating Processes $\frac{1}{s}$

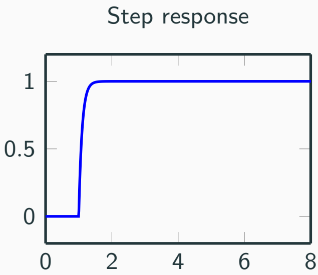


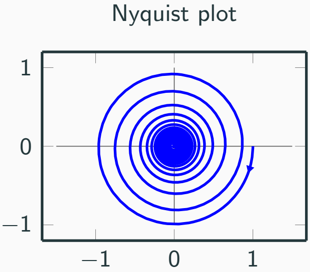
Oscillative Processes

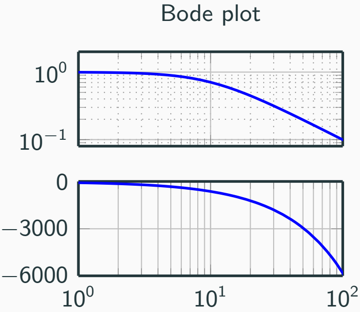
$$\frac{K\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2},\ 0 < \zeta < 1$$



Delay Processes $\frac{K}{sT+1}e^{-sL}$

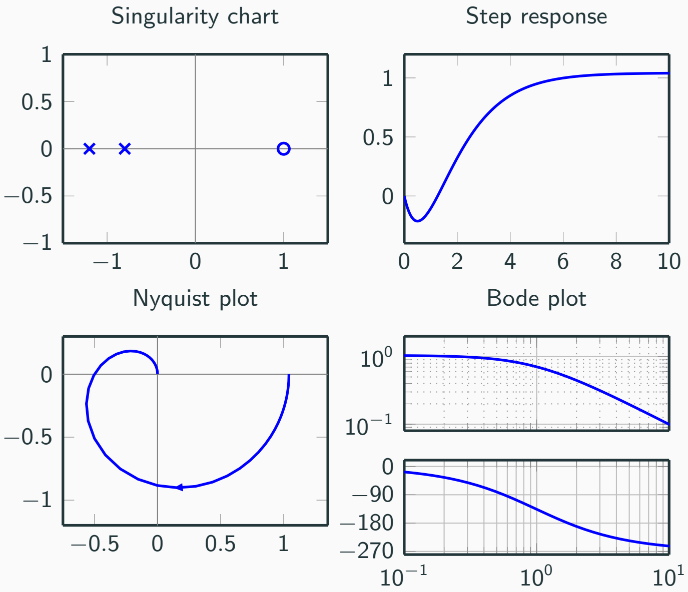






Process with Inverse Responses

$$\frac{-sa+1}{(sT_1+1)(sT_2+1)}$$



Content

This lecture

1. Step Response Analysis

2. Frequency Response

3. Relation between Model Descriptions

Next lecture

- Classic Feedback Example - The Steam Engine
- Stability
- Stationary Errors