Lund University

Department of
AUTOMATIC CONTROL

Automatic Control, Basic Course

Exam 19 mars 2019, 8:00-13:00

Points and grades

All solutions must be well motivated. The whole exam gives 25 points. The number of points are presented after each problem. Preliminary grades:

Grade 3: at least 12 points,
4: at least 17 points,
5: at least 22 points.

Aids

Mathematical collections of formulae (e.g. TEFYMA), collections of formulae in automatic control, and calculators that are not programmed in advance.

Results

The results are presented through LADOK. Time and place for exam presentation will be announced on the course web page.

Lycka till!

Lösningar till tentamen i Reglerteknik AK

1. A system is described by the following differential equation:

$$\ddot{y} + 8\dot{y} + 16y = \dot{u} - 3u$$

a. What is the transfer function of the system from u to y? (1 p)

b. What are the poles and zeros of the system? Is it asymptotically stable? Remember to give justification for your answers. (1 p)

c. The process is regulated by a proportional controller with gain K in a closed loop system (Figure 1). For what values of K is the closed loop system asymptotically stable? Consider both positive and negative values of K. (2 p)

d. Suppose that we make a unit step change in setpoint r. Determine the stationary control error e. How small error can we get using the P controller? (2 p)

e. What control structure could guarantee zero stationary error? (1 p)

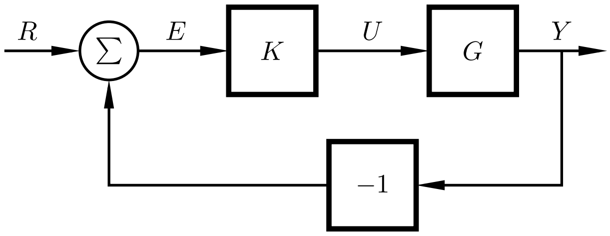

Figure 1: Closed loop system in Problem 1.c.

Solution

a. Performing a Laplace Transform on the differential equation, the transfer function of the system is given by $Y(s)(s^2 + 8s + 16) = U(s)(s - 3)$:

$$\begin{aligned} G(s) &= \frac{Y(s)}{U(s)} \\ &= \frac{s-3}{s^2+8s+16} \\ &= \frac{s-3}{(s+4)(s+4)} \end{aligned}$$

b. Zeros: $s = 3$
Poles: $s = -4, -4$

Both poles have negative real parts, therefore the system is asymptotically stable.

c. The transfer function of the closed loop system is $G_{cl}(s) = Y(s)/R(s) = KG(s)/(1 + KG(s))$:

$$
\begin{aligned}
G_{cl}(s) &= \frac{KG(s)}{(1 + KG(s))} \\
&= \frac{K(s-3)}{(s+4)^2 + K(s-3)} \\
&= \frac{K(s-3)}{s^2 + s(8+K) + (16-3K)}
\end{aligned}
$$

The system is asymptotically stable when both poles are strictly in the left-half plane (have negative real parts). This is true if and only if $a_1, a_2 > 0$ for the characteristic polynomial $s^2 + a_1 s + a_2$. Therefore,

$$
\begin{aligned}
8 + K &> 0 \\
K &> -8
\end{aligned}
$$

$$
\begin{aligned}
16 - 3K &> 0 \\
16 &> 3K \\
-8 < K &< \frac{16}{3}
\end{aligned}
$$

d. The static error of the response to a unit step can be computed by applying the final value theorem to the transform of the error in response to a step input. The transfer function from the reference to the error is:

$$
G_{re}(s) = \frac{(s+4)^2}{(s+4)^2 + K(s-3)}
$$

We then apply the final value theorem. Skipping some theoretical steps this is equivalent to evaluate $G_{re}(s)$ in $s = 0$.

$$
G_{re}(0) = \frac{16}{16 - 3K}
$$

We want therefore to maximize the module of the denominator (within the stability range). This is obtained with K as small as possible in the interval, i.e., as close to -8 as possible (without being -8). This gives a static error of approximately $e(\infty) = 16/40 = 0.4$.

e. An integral controller would guarantee zero stationary error, given the closed loop system is asymptotically stable.

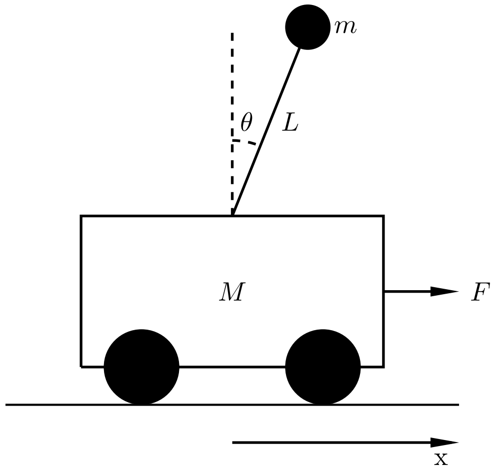

Figure 2: An inverted pendulum, held up by applying a force to the wagon.

2. A simplified model of an inverted pendulum (see Figure 2) is described by the following equations

$$\ddot{x} = \frac{-mL\sin(\theta)\dot{\theta}^2 + F + mg\cos(\theta)\sin(\theta)}{M + m\sin(\theta)^2}$$
$$\ddot{\theta} = \frac{mL\cos(\theta)\sin(\theta)\dot{\theta}^2 + F\cos(\theta) + (M+m)g\sin(\theta)}{L\cdot(M + m\sin(\theta)^2)}$$

a. Write the system in state-space form using the state vector $\mathbf{x} = (x, \dot{x}, \theta, \dot{\theta})^T$ and control signal $u = F$. (0.5 p)

b. $\mathbf{x}^0 = (0, 0, 0, 0)^T$ and $u^0 = 0$ is a suitable point around which to linearize the system. Explain why this is an appropriate choice. (0.5 p)

c. Linearize the system around the chosen point, and express your answer as the system matrices A and B. (2 p)

Solution

a. We use $\mathbf{x} = (x, \dot{x}, \theta, \dot{\theta})^T$ and $u = F$ which gives us

$$\dot{x_1} = x_2$$
$$\dot{x_2} = \frac{-mL\sin(x_3)x_4^2 + u + mg\cos(x_3)\sin(x_3)}{M + m\sin(x_3)^2}$$
$$\dot{x_3} = x_4$$
$$\dot{x_4} = \frac{mL\cos(x_3)\sin(x_3)x_4^2 + u\cos(x_3) + (M+m)g\sin(x_3)}{L\cdot(M + m\sin(x_3)^2)}$$

b. Firstly, it is a stationary point, which is simple to control.

$$
\begin{aligned}
\dot{x_1} &= x_2 = 0 \\
\dot{x_2} &= \frac{-mL\sin(0)0^2 + 0 + mg\cos(0)\sin(0)}{M + m\sin(0)^2} = 0 \\
\dot{x_3} &= x_4 = 0 \\
\dot{x_4} &= \frac{mL\cos(0)\sin(0)0^2 + 0\cos(0) + (M+m)g\sin(0)}{L\cdot(M + m\sin(0)^2)} = 0
\end{aligned}
$$

There are other stationary points, but we want to choose one that is near the desired state of the system to achieve good control at that point. The desired state is a stationary wagon with the pendulum straight up, that is $x_2 = x_3 = x_4 = u = 0$. The position, x_1, is not relevant to this desired state so 0 is also a suitable choice.

c. Differentiation gives

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{ML} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix}
$$

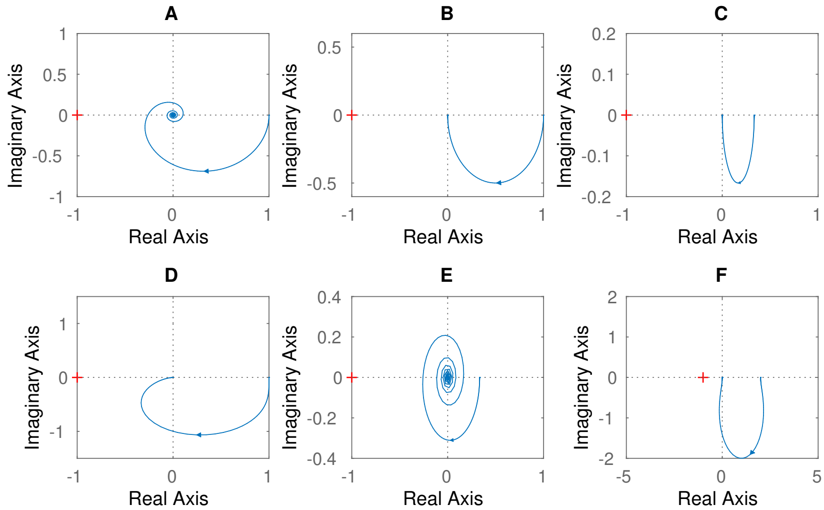

Figure 3: Nyquist diagrams for Problem 3.

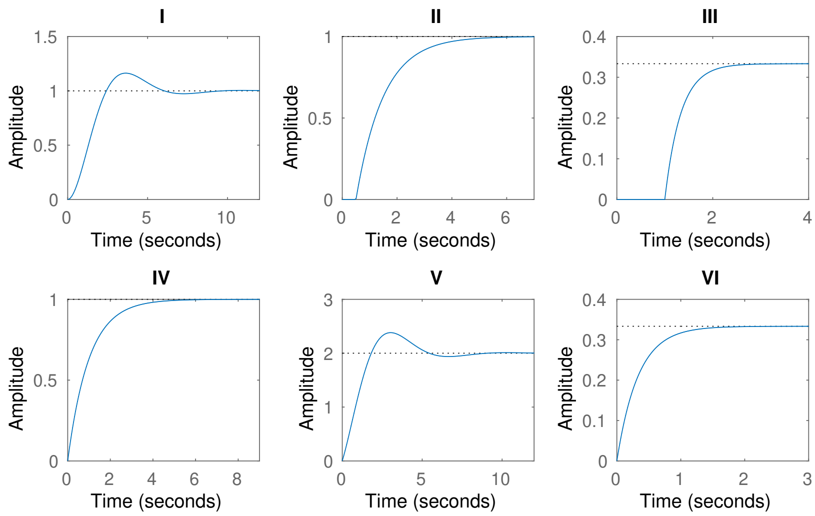

Figure 4: Step responses for Problem 3.

3. Match each transfer function with one Nyquist diagram and one step response. Motivate each choice. (3 p)

$$G_1(s) = \frac{1}{s+1}$$
$$G_2(s) = \frac{e^{-s}}{s+3}$$
$$G_3(s) = \frac{s+2}{s^2+s+1}$$

Solution

G_1 is a first order process without delay and a static gain of 1. Only step re-

sponse IV and Nyquist diagram B fit these properties.
G_2 is a first order process with a delay of 1 second and a static gain of 1/3. Only step response III and Nyquist diagram E fit these properties.
G_3 is a second order system with a static gain of 2 and one zero. Only step response V fits these properties and Nyquist diagram F is the only one with a static gain of 2.

G_1 - IV - B
G_2 - III - E
G_3 - V - F

4. Consider the following statements based on the system $G(s)$ below and determine whether they are true or false. Justify your answers.

$$G(s) = \frac{s+2}{(s+1)(s+3)}$$

a. $G(s)$ is a first order system. (0.5 p)

b. The given information is enough to determine if $G(s)$ is an asymptotically stable system. (0.5 p)

c. If we apply the control signal $u(t) = \sin(3t)$ to the system, the output in stationarity will be a sinusoid with double the frequency, i.e. $u(t) = A\sin(6t + \varphi)$. (0.5 p)

d. Feedback with a PI controller allows us to place the closed-loop system's poles whereever we want by calibrating the PI controller's parameters. (1.5 p)

Solution

a. False, it is a second order system (two poles).

b. True, it is enough. The system is asymptotically stable since all poles lie strictly in the left half plane.

c. False, amplitude and phase can be affected, but frequency will remain the same.

d. False, the system has two poles, and a PI controller has only one, which gives three poles in the closed-loop system. However, we have only two parameters (i.e. K and Ti) and are therefore unable to control 3 poles. The denominator of the closed-loop system's transfer function (characteristic polynomial) becomes

$$s^3 + (4+K)s^2 + (\frac{1}{T_i} + 2K + 4)s + 2K$$

which also illustrates this clearly. Since both the constant and quadratic term depend only on K, we will never be able to control both of them as we wish.

5. Consider the Bode diagram of an open-loop transfer function shown in 5 and answer the following questions.

a. Determine the open-loop transfer function of the system. (1 p)

b. Determine the phase and amplitude margin of the system. (1 p)

c. The system does not have very much phase margin. Introduce a compensation link that makes the system's phase margin 50° without changing its cross-over frequency. (2 p)

d. How much time delay can the compensated system handle before it becomes unstable? (1 p)

Solution

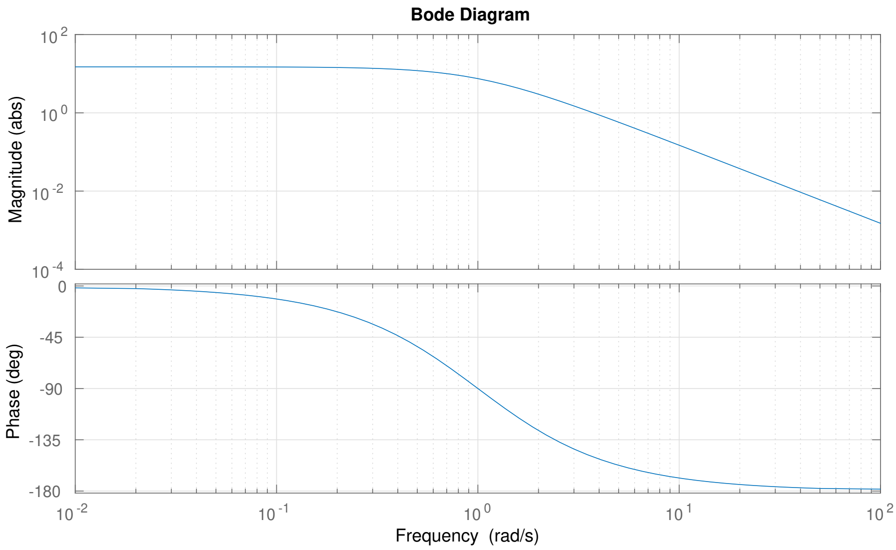

Figure 5: Bode diagram of the open-loop transfer function in Problem 5.

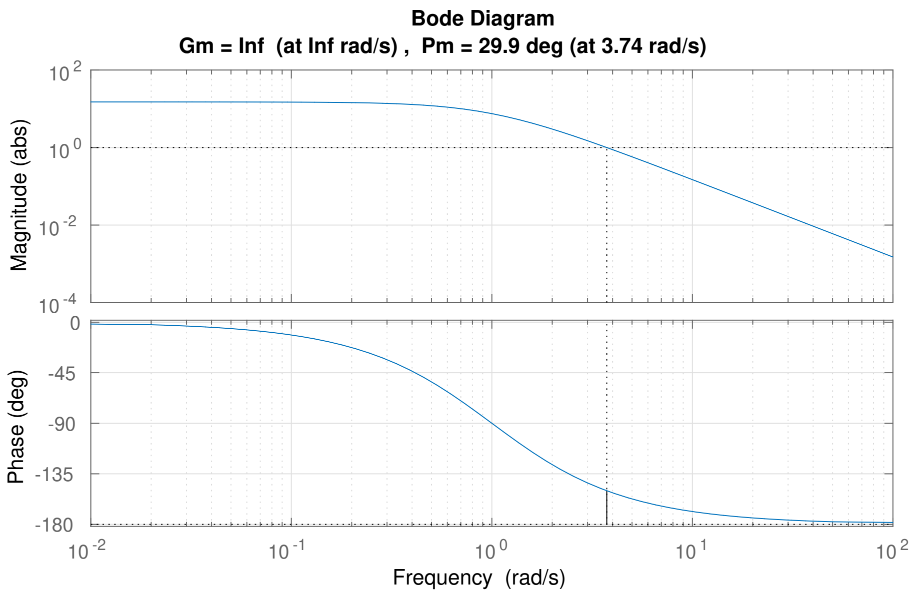

Figure 6: Margins for Problem 5.

a. Two poles at $s = -1$, shown by a the corner frequency at 1 rad/s and downward slope of 10^2 per decade. A static gain of approx. 15.

$$G(s) = \frac{15}{(s+1)^2}$$

b. In figure 6 we see the margins. We get $\varphi_m(i\omega_c) \approx 30$ for $\omega_c \approx 3.7$ and $A_m = \infty$ since we never cross the negative real axis (i.e. the phase asymptotically approaches -180 degrees).

c. Our objective is to increase the phase margin, so we choose a lead compensation link. We have $\omega_c = 3.7$ and want $\hat{\varphi}_m(i\omega_c) = 50$ which means that $\Delta\varphi = 50 -$

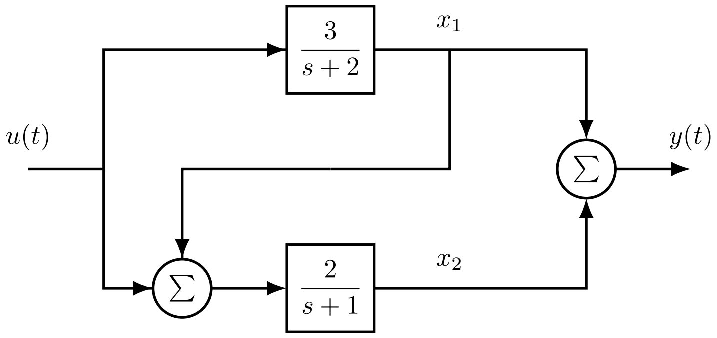

Figure 7: Block diagram for Problem 6.

$30 = 20$. Reading from the formula-sheet gives $N = 2$ for $\Delta\varphi = 20$ and we want the maximum phase increase at the cross-over frequency, so $\omega = \omega_c$ which gives $b = \omega/\sqrt{N} = 2.6$. We want to maintain the same cross-over frequency which means that unit gain occurs at the cross-over frequency. From the formula-sheet we see that gain is equal to $K_K\sqrt{N}$, which gives $K_K = 1/\sqrt{N}$. Therefore, our complete lead compensation link

$$G_K(s) = 1.4\frac{s + 2.6}{s + 5.2} = 0.7\frac{1 + s/2.6}{1 + s/5.2}$$

d. Lag margin for a system is given by $L = \varphi_m/\omega_c = 0.24$ where $\varphi_m = 50\ ° = 0.87$ rad and $\omega_c = 3.7$ rad/s. This means that the system can withstand a dealy of 0.24 s before it reaches the stability boundary.

6. Study the process described by the block diagram in Figure 7. A state-feedback link was created for the process that places the poles in the left half plane with frequency (distance from origin) of $\omega_0 = 6$ rad/s and angle of 45° from the negative real axis. Unfortunately, it became apparent that only some of the system states could be observed. You have now been hired to create a Kalman filter to estimate the states so that the state-feedback link can be used.

a. Write the system in state-space form using the states x_1 and x_2 given in the block diagram. Verify that it is an observable system. (2 p)

b. Design a suitable Kalman filter for this process and state-feedback link. (2 p)

Solution

a. Begin by writing each state in the Laplace domain, then do the inverse Laplace transform to express each state as a differential equation. Then, rewrite the system of equations in the A, B, C, D matrix form

$$
\begin{aligned}
X_1 &= \frac{3}{s+2}U \\
\dot{x}_1 &= -2x_1 + 3u \\
X_2 &= \frac{2}{s+1}(X_1 + U) \\
\dot{x}_2 &= 2x_1 - x_2 + 2u \\
Y &= X_1 + X_2 \\
y &= x_1 + x_2 \\
\dot{x} &= \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
y &= \begin{bmatrix} 1 & 1 \end{bmatrix}
\end{aligned}
$$

To determine whether the system is observable we check that the observability matrix has a nonzero determinant.

$$
\begin{aligned}
W_o &= \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\
\det(W_o) &\neq 0
\end{aligned}
$$

b. Introduce $\hat{x}$ as our state estimate in the Kalman filter and introduce the estimation error term $K(y - C\hat{x})$.

$$
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) \\
\dot{\hat{x}} &= (A - KC)\hat{x} + Bu + Ky \\
s\hat{X} &= (A - KC)\hat{X} + BU + KY \\
\hat{X} &= (sI - A + KC)^{-1}(BU + KY)
\end{aligned}
$$

From this, we want to choose K for the desired pole placement. A rule of thumb is to have poles that are twice as fast as the state-feedback, so we choose poles with $\omega_0 = 12$ and $\zeta = \cos 45 = 1/\sqrt{2}$ which gives the characteristic polynomial $s^2 + 12\sqrt{2}s + 144$.

$$
\begin{aligned}
\det(sI - A + KC) &= \begin{vmatrix} s + 2 + k_1 & k_1 \\ k_2 - 2 & s + 1 + k_2 \end{vmatrix} \\
&= (s + 2 + k_1)(s + 1 + k_2) - k_1(k_2 - 2) \\
&= s^2 + (3 + k_1 + k_2)s + 2 + 3k_1 + 2k_2
\end{aligned}
$$

Comparing this to the desired characteristic polynomial gives the equations

$$
\begin{cases} 3 + k_1 + k_2 = 12\sqrt{2} \\ 2 + 3k_1 + 2k_2 = 144 \end{cases} \implies \begin{cases} k_1 = 148 - 24\sqrt{2} \\ k_2 = 36\sqrt{2} - 151 \end{cases}
$$