LUNDS UNIVERSITET

Department of
AUTOMATIC CONTROL

Exam FRTF01 - Physiological Models and Computation

January 13 2015, 14-19

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Preliminary grades:

Grade 3: 12–16.5 points
4: 17–21.5 points
5: 22–25 points

Accepted aid

Lecture slides, any books (without relevant exercises with solutions), standard mathematical tables and "Formelsamling i reglerteknik". Calculator.

Results

The result of the exam will be posted in LADOK no later than January 20. Information on when the corrected exam papers will be shown, will be given on the course homepage.

1. Consider the following model that describes digestion

$$
\begin{aligned}
\dot{q}_{\text{sto1}} &= -k_1 \cdot q_{\text{sto1}} + u \\
\dot{q}_{\text{sto2}} &= k_1 \cdot q_{\text{sto1}} - k_2 \cdot q_{\text{sto2}} \\
\dot{q}_g &= k_2 \cdot q_{\text{sto2}} - k_3 \cdot q_g \\
y &= \frac{q_g}{V_g}
\end{aligned}
$$

where u is the amount of ingested carbohydrates per time unit, q_{sto1} is the amount of glucose in the solid stomach compartment, q_{sto2} is the amount of glucose in the liquid stomach compartment, q_g is the glucose mass in the intestine and y is the measured concentration of glucose in the gut. Model parameters k_1, k_2, k_3 and V_g are all positive and constant and described in Table 1.

Table 1: Description of model parameters in Problem 1

Parameter	Description
k_1	rate of grinding, min^{-1}
k_2	rate of gastric emptying, min^{-1}
k_3	rate of intestinal absorption, min^{-1}
V_g	volume of gut compartment, l

a. Draw a schematic of the compartment model. (1 p)

b. The values of the rate constants differ between different types of meals. Suppose that a meal A is digested faster than another meal B. Which parameter(s) would you expect to change and how between meal A and B? (0.5 p)

c. Assume that k_2 and k_3 are known. Is k_1 identifiable? Motivate your answer. (1 p)

d. Let $u(t) = 0$. Then, for a specific type of meal, the half-life of q_{sto1} is 30 min. Determine k_1. (1 p)

e. The remaining parameters have the following values; $k_2 = 0.02$ min^{-1}, $k_3 = 0.5$ min^{-1} and $V_g = 1$ l. Determine which of the two step responses (1 or 2) shown in Figure 1 that is given by the system. Motivate your answer. (0.5 p)

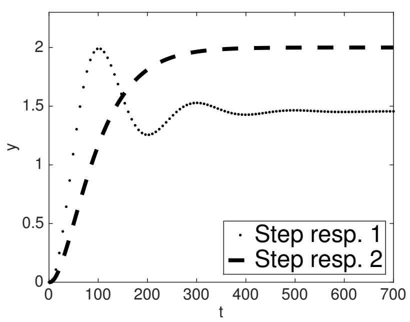
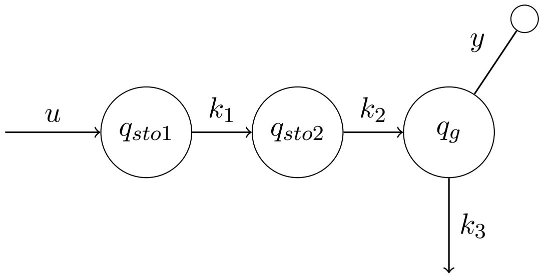


Figure 1: Step responses for Problem 1, subproblem d.

Solution

a. A schematic of the compartmental model is given below:



b. Digestion is faster if parameters k_1, k_2 and k_3 are larger.

c. The transfer function of the system from u to y is

$$G_{yu} = \frac{1}{V_g}\frac{k_1 \cdot k_2}{(s+k_1)(s+k_2)(s+k_3)}.$$

If k_2 and k_3 are known, it is possible to identify k_1. This can, for instance, be done by first determining V_g from the static gain, and then identifying k_1 by letting the input signal be a sinus with a fixed frequency.

d. With $u(t) = 0$, the differential equation becomes

$$\dot{q}_{\text{sto1}} = -k_1 \cdot q_{\text{sto1}},$$

and the solution is

$$q_{\text{sto1}}(t) = C_0 e^{-k_1 t},$$

where C_0 is a constant. Given that the half-life is 30 min, we get

$$\frac{1}{2}C_0 = C_0 e^{-k_1 \cdot 30}$$

which yields

$$k_1 = \frac{\log 2}{30} \approx 0.02 \text{ min}^{-1}.$$

e. The static gain of the system is given by

$$G_{yu}(0) = \frac{1}{V_g}\frac{1}{k_3} = 2.$$

The step response corresponding to this static gain is Step response 2, given by the dashed line. Furthermore, the system has no imaginary poles. Thus, there can not be any oscillations (which further rules out Step response 1).

2. The dynamics of a specific system is given by the following non-linear differential equation

$$\ddot{z} + \frac{\dot{z}^4}{z^2} - z = \sqrt{u+1},$$

where u is the input signal and the output signal is given by $y = z^2 + u^2$.

a. Introduce states $x_1 = z$ and $x_2 = \dot{z}$ and find all stationary points (x_1^0, x_2^0, u^0, y^0). (1.5 p)

b. Linearize the system around the stationary point corresponding to $u^0 = 3$. (1.5 p)

Solution

a.

$$\begin{aligned} \dot{x}_1 &= x_2 && (= f_1(x,u)) \\ \dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} && (= f_2(x,u)) \\ y &= x_1^2 + u^2 && (= g(x,u)) \end{aligned} \tag{1}$$

Thus, from the first equation in (1) we get $x_2^0 = 0$. Further, $x_2 = 0$ inserted in the second equation gives

$$0 = x_1 + \sqrt{u+1}. \tag{2}$$

The stationary points are $(x_1^0, x_2^0, u^0) = (-\sqrt{t+1}, 0, t)$, where $t \geq -1$. In stationarity the output signal is given by $y^0 = t + 1 + t^2$.

b. $u^0 = 3$ gives the stationary point $(x_1^0, x_2^0, u^0, y^0) = (-2, 0, 3, 13)$. The partial derivatives are

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} = 0, && \frac{\partial f_1}{\partial x_2} = 1, && \frac{\partial f_1}{\partial u} = 0, \\ &\frac{\partial f_2}{\partial x_1} = 2\frac{x_2^4}{x_1^3} + 1, && \frac{\partial f_2}{\partial x_2} = -4\frac{x_2^3}{x_1^2}, && \frac{\partial f_2}{\partial u} = \frac{1}{2\sqrt{u+1}}, \\ &\frac{\partial g}{\partial x_1} = 2x_1, && \frac{\partial g}{\partial x_2} = 0, && \frac{\partial g}{\partial u} = 2u. \end{aligned}$$

Introduce new variables

$$\begin{aligned} \Delta x &= x - x^0, \\ \Delta u &= u - u^0, \\ \Delta y &= y - y^0. \end{aligned} \tag{3}$$

The linearized system is given by

$$\begin{aligned} \dot{\Delta x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} \Delta u \\ \Delta y &= \begin{bmatrix} -4 & 0 \end{bmatrix} \Delta x + 6\Delta u \end{aligned} \tag{4}$$

3. Assume that you have two neurons described by the following differential equations

$$\begin{aligned} C_m \frac{dV_1}{dt} &= -g_1(V_1 - E_1) + u_1 \\ C_m \frac{dV_2}{dt} &= -g_2(V_2 - E_2) + u_2 \end{aligned}$$

where V_1 is the membrane potential of the first neuron, V_2 is the membrane potential of the second neuron and u_i, $i = 1$ and 2, are external input currents. The parameters of the model are given in Table 2.

Table 2: Description of parameters in problem 3

Parameter	Description	Value
C_m	membrane capacitance	1 [μF/cm^2]
g_1	membrane conductance of neuron 1	0.5 [mS/cm^2]
E_1	equilibrium potential of neuron 1	-60 mV
g_2	membrane conductance of neuron 2	0.8 [mS/cm^2]
E_2	equilibrium potential of neuron 2	-60 mV

a. Perform a variable transformation $\Delta V_1 = V_1 - E_1$ and $\Delta V_2 = V_2 - E_2$ and write down the new system of equations. (1 p)

b. Connect the two neurons in line $1 \to 2$ with the input to neuron 2 being $u_2 = \Delta V_1/2$ and the input to neuron 1 being $u_1 = I_{ext}$. Assume that you can measure the membrane potential of neuron 2, i.e., $y = \Delta V_2$. Write the system on state-space form. (1 p)

c. Determine the amplitude of the oscillations of the membrane potential of neuron 2 when $I_{ext}(t) = 3\sin 2t$. (1.5 p)

Solution

a. Since E_1 and E_2 are constants, the new system of equations becomes

$$\begin{aligned} C_m \frac{d\Delta V_1}{dt} &= -g_1 \cdot \Delta V_1 + u_1 \\ C_m \frac{d\Delta V_2}{dt} &= -g_2 \cdot \Delta V_2 + u_2. \end{aligned}$$

b. With $u_1 = I_{ext}$, $u_2 = \Delta V_1/2$ and $y = \Delta V_2$, the new system of equations becomes

$$\begin{aligned} C_m \frac{d\Delta V_1}{dt} &= -g_1 \cdot \Delta V_1 + I_{ext} \\ C_m \frac{d\Delta V_2}{dt} &= -g_2 \cdot \Delta V_2 + \Delta V_1/2 \\ y &= \Delta V_2. \end{aligned}$$

The state-space representation with state vector $\Delta V = [\,\Delta V_1 \quad \Delta V_2\,]^T$ is then

$$\begin{aligned} \Delta\dot{V} &= \underbrace{\frac{1}{C_m}\begin{pmatrix} -g_1 & 0 \\ 0.5 & -g_2 \end{pmatrix}}_{A} \Delta V + \underbrace{\frac{1}{C_m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B} I_{ext} \\ y &= \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{C} \Delta V. \end{aligned}$$

c. With matrices A, B and C defined in the previous subproblem, the transfer function is given by

$$
\begin{aligned}
G(s) = C(sI-A)^{-1}B &= \frac{1}{C_m}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} s+g_1 & 0 \\ -0.5 & s+g_2 \end{pmatrix}^{-1}\frac{1}{C_m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{(C_m)^2(s+g_1)(s+g_2)}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} s+g_2 & 0 \\ 0.5 & s+g_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{0.5}{(C_m)^2(s+g_1)(s+g_2)}.
\end{aligned}
$$

Now,

$$
|G(iw)| = \frac{0.5}{(C_m)^2\sqrt{w^2+g_1^2}\sqrt{w^2+g_2^2}}
$$

and with $w = 2$, $g_1 = 0.5$, $g_2 = 0.8$ and $C_m = 1$ we get

$$
|G(2i)| = \frac{0.5}{\sqrt{4.25}\sqrt{4.64}} = 0.11.
$$

The amplitude of the oscillations is then $3 \cdot 0.11 = 0.33$, as $I_{ext}(t) = 3\sin(2t)$.

4 a. Consider the system

$$
G_P(s) = \frac{s+1}{s(s-4)}
$$

in negative feedback interconnection with a P-controller $G_C(s) = K$. For which values of K does the closed-loop system become asymptotically stable? (1 p)

b. Consider the system

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} -1 & 0 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 4 \\ -2 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & -2 \end{pmatrix} x.
\end{aligned}
$$

Determine L and l_r in the state-feedback control law

$$
u(t) = -Lx + l_r r
$$

such that both poles of the system are placed in -2 and that the stationary gain is 1. (2 p)

c. The following system

$$
G_P = \frac{1}{(s+0.1)^2}
$$

is to be controlled by a P-controller. You get to choose between two different P-controllers (1 or 2). The sensitivity functions given the two controllers are given in Figure 2. The criteria are that disturbances with frequencies below 0.1 rad/s should be attenuated and no disturbances should be amplified by more than a factor of 10. Motivate your answer. (1 p)

Solution

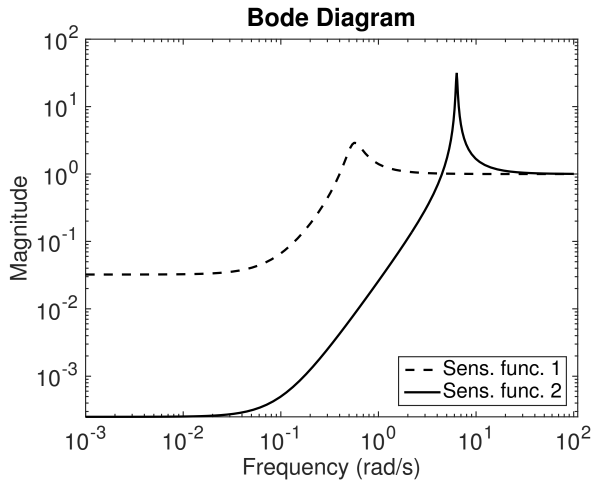


Figure 2: Bode Diagram for Problem 4, subproblem c.

a. The transfer function for the closed loop system is

$$G_{cl}(s) = \frac{G_C(s)G_P(s)}{1+G_C(s)G_P(s)} = \frac{K(s+1)}{s(s-4)+K(s+1)} = \frac{K(s+1)}{s^2+(K-4)s+K}.$$

The system is asymptotically stable if

$$(K-4) > 0 \quad \text{and} \quad K > 0.$$

Hence it is stable if $K > 4$.

b. Let $L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}$, then with the proposed feedback law, we get that

$$\dot{x} = \overbrace{\begin{pmatrix} -1-4l_1 & -4l_2 \\ -1+2l_1 & -2+2l_2 \end{pmatrix}}^{\tilde{A}} x + \begin{pmatrix} 4 \\ -2 \end{pmatrix} l_r r,$$

and the system has the characteristic polynomial

$$\begin{aligned} \det(sI-\tilde{A}) &= (s+1+4l_1)(s+2-2l_2) + 4l_2(-1+2l_1) \\ &= s^2 + (3+4l_1-2l_2)s + 8l_1 - 6l_2 + 2. \end{aligned}$$

We want the system to have both poles in -2, i.e., the characteristic polynomial should be $(s+2)^2 = s^2+4s+4$. Matching the coefficients yields

$$\begin{aligned} 3+4l_1-2l_2 &= 4 \\ 8l_1-6l_2+2 &= 4 \end{aligned}$$

and hence $l_1 = 1/4$ and $l_2 = 0$. Finally, the static gain is given by

$$\begin{aligned} G(0) = C(0\cdot I-\tilde{A})^{-1}Bl_r &= \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ \frac{1}{2} & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ -2 \end{pmatrix} l_r \\ &= \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{-1}{8} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} l_r = 5l_r. \end{aligned}$$

Hence $l_r = 1/5$.

c. Sensitivity function 2 amplifies a disturbance with a frequency around 10 rad/s more than 10 times. Hence, it is only sensitivity function 1 that fulfils the specifications.

5. Assume that the concentration of bacteria, x, in a jam pot is decreasing with a constant rate a times the concentration and increasing with a constant rate b times the concentration squared.

a. Write down the differential equation of the concentration of bacteria x. (0.5 p)

b. Estimate the model parameters a and b given the measurements in Table 3. (1.5 p)

Table 3: Measurements in problem 5

Concentration of bacteria	Rate of change
0.1	-0.1
1	-3.5
1.5	-0.4
2	3.6
3	11

Solution

a.

$$\dot{x} = -ax + bx^2$$

b. Denote $y = \dot{x}$. Then,

$$\mathcal{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \underbrace{\begin{pmatrix} -x_1 & x_1^2 \\ -x_2 & x_2^2 \\ -x_3 & x_3^2 \\ -x_4 & x_4^2 \\ -x_5 & x_5^2 \end{pmatrix}}_{\Phi} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

where $y_1, ..., y_5$ are the measurements given under *Rate of change* and $x_1, ..., x_5$ are the measurements given under *Concentration of bacteria*, in Table 3. The estimates are then determined as follows

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = (\Phi^T\Phi)^{-1}\Phi^T\mathcal{Y} = \begin{pmatrix} 4.6 \\ 2.8 \end{pmatrix}.$$

6. Three systems, G_1, G_2 and G_3, are connected as shown in Figure 3.

a. Determine the transfer function from u to y. (1 p)

b. Determine the pole(s) of the system when the subsystems are given by

$$G_1(s) = \frac{1}{s+3}, \qquad G_2(s) = 3 \qquad \text{and} \qquad G_3(s) = \frac{1}{s+4}.$$

Is the system stable? (1.5 p)

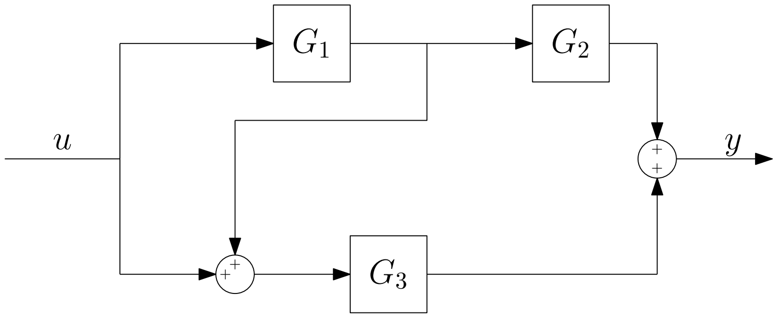


Figure 3: Figure to Problem 6.

c. Determine and sketch the step response of the system and mark the value of the static gain by a dashed line. (1.5 p)

Solution

a. The transfer function from u to y is given by

$$G_{yu} = G_3 + G_2G_1 + G_3G_1.$$

b. With the given subsystems, the transfer function becomes

$$G_{yu} = \frac{1}{s+4} + \frac{3}{s+3} + \frac{1}{(s+3)(s+4)} = \frac{4}{s+3}.$$

The system has a pole in -3. Notice the pole-zero cancellation of the pole in -4.

c. When the input signal is a step, $U(s) = 1/s$, the output signal is given by

$$Y(s) = \frac{4}{s+3} \cdot \frac{1}{s}.$$

Inverse Laplace transformation gives

$$\mathcal{L}^{-1}(Y(s)) = \frac{4}{3}(1 - \mathrm{e}^{-3t})$$

as can be seen by (14) in the collection of formulae. The step response of the system is given in Figure 4, black line, with static gain marked by dashed red line.

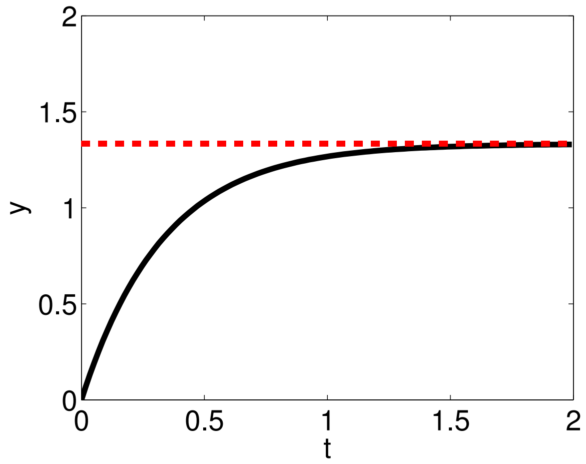


Figure 4: Step response for the system in Problem 6 in black, and static gain marked by dashed red line.

7. The following differential equations describe the dynamics of the epidemiological model denoted the SIS-model

$$\frac{dS}{dt} = -\beta IS + \gamma I$$
$$\frac{dI}{dt} = \beta IS - \gamma I$$

where $S(t)$ is the fraction of susceptible people and $I(t)$ is the fraction of infected people, at time $t \geq 0$. Furthermore, $\beta > 0$ and $\gamma > 0$ are positive constant parameters and the initial state is given by $S(0) = 0.8$ and $I(0) = 0.2$.

a. Show that $S(t) + I(t) = 1$ for all times $t \geq 0$. (1 p)

b. Find the stationary points of the SIS-model, denoted $\left(S^0, I^0\right)$. (1.5 p)

c. Explain qualitatively (i.e., in words) what happens with the system if $\beta = 0$, given the initial condition stated above. (1 p)

d. In Figure 5, the phase portrait of the dynamics described above is shown together with four trajectories. Decide which of the four trajectories (A, B, C or D) that is the correct one, given the initial state. Do not forget to motivate your answer. (1 p)

Solution

a. Using the fact that

$$\frac{dS}{dt} + \frac{dI}{dt} = -\beta IS + \gamma I + \beta IS - \gamma I = 0,$$

it follows that

$$S(t) + I(t) = C,$$

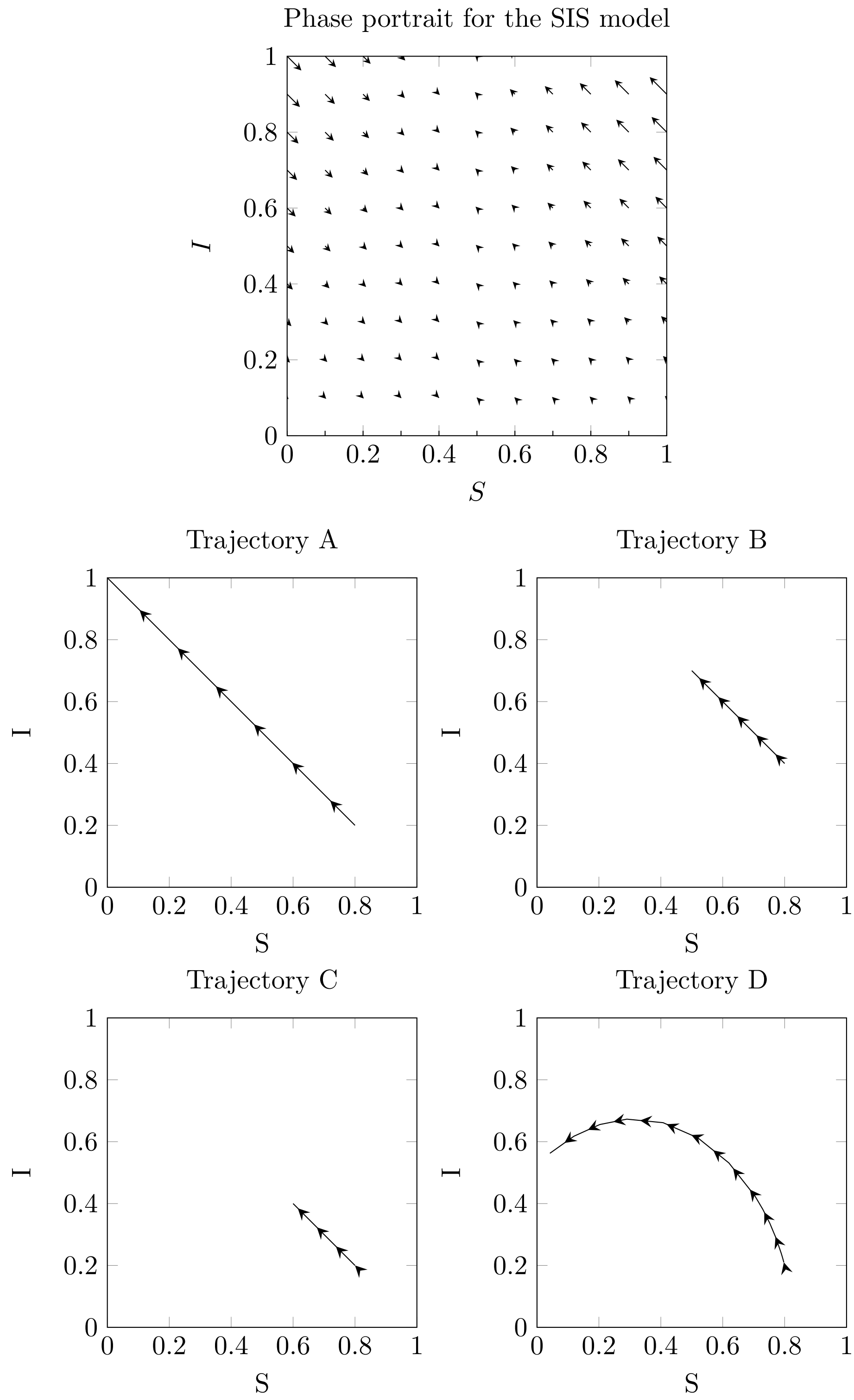


Figure 5: The phase portrait and the trajectories for Problem 7.

where C is a constant. We know that for time $t = 0$

$$C = S(0) + I(0) = 0.8 + 0.2 = 1$$

and thus, $S(t) + I(t) = 1$.

b.

$$\frac{dS}{dt} = -\beta IS + \gamma I = 0$$
$$\frac{dI}{dt} = \beta IS - \gamma I = 0$$

gives that $I(\beta S - \gamma) = 0$. Using the fact that $I(t) + S(t) = 1$ for all t, two stationary points can be derived

$$\left(S^0, I^0\right)_1 = (1, 0) \text{ and } \left(S^0, I^0\right)_2 = (\gamma/\beta, 1 - \gamma/\beta).$$

c. If $\beta = 0$ the model becomes

$$\frac{dS}{dt} = \gamma I$$
$$\frac{dI}{dt} = -\gamma I.$$

Given that the system starts in $(S(0), I(0)) = (0.8, 0.2)$ the fraction of infected people will decrease from 0.2 to 0 while the fraction of susceptible people will increase from 0.8 to 1 (as $S(t) + I(t) = 1$ for all t). More specifically, the infected people will get susceptible and no susceptible people will get infected, i.e., there is only a flow from compartment I to compartment S as $\gamma > 0$.

d. The trajectory should start in $(0.8, 0.2)$, so it can not be B. Moreover, it can be seen from the phase portrait that all trajectories will be straight lines, so it can not be D either. Trajectory A is going in the opposite direction to the phase portrait for $S < 0.5$, so the correct trajectory is C.

Good Luck!