Exercise for Optimal control – Week 6

Choose **1.5** problems to solve.

Exercise 1. Derive the policy iteration scheme for the LQR problem

$$\min_{u(\cdot)} \sum_{k=1}^{\infty} x_k^\top Q x_k + u_k^\top R u_k$$

with $Q = Q^\top \geq 0$ and $R = R^\top > 0$ subject to:

$$x_{k+1} = Ax_k + Bu_k.$$

Assume the system is stabilizable. Start the iteration with a stabilizing policy. Run the policy iteration and value iteration on a computer for the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = I$$

Compare the convergence rates of the two iterations scheme for the policies and value functions.

Exercise 2 (LQR for LTV systems)**.** Consider a controllable LTV system

$$\dot{x} = A(t)x + B(t)u$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and cost function

$$J = x(T)^\top Q_f x(T) + \int_{t_0}^{T} x(t)^\top Q(t) x(t) + u(t)^\top R(t) u(t) \mathrm{d}t$$

where Q_f, $Q(t) \geq 0$ and $R(t) > 0$ for all $t \geq 0$. In addition, we assume $A(\cdot)$, $B(\cdot)$, $Q(\cdot)$ and $R(\cdot)$ are continuous. The objective is to find an optimal control u^* such that J is minimized.

1) The dynamic programming works also for time varying systems. Write down the Hamiltonian $H(t, x, u, p)$ for this problem and derive the optimal controller using the verification rule. *Hint:* consider value function of the form $J^*(t, x) = x^\top P(t) x$.

2) Show that the HJB equation reduces to an ODE:

$$-\dot{P}(t) = Q(t) + P(t)A(t) + A(t)^\top P(t) - P(t)B(t)R(t)^{-1}B(t)^\top P(t). \tag{1}$$

with boundary condition

$$P(T) = Q_f.$$

3) Prove that the equation (1) has a unique symmetric semi-positive definite solution on interval $[0, T]$ for any $T > 0$. In particular, there is no finite escape time.

Example 3. 1) Derive the HJB equation for the time optimal control problem of the double integrator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

with initial condition $(1, 1)$ and terminal condition $(0, 0)$ under the constraint $|u| \leq 1$.

2) Solve the HJB equation using method of characteristics.