Exercise for Optimal control – Week 5

Choose **2** problems to solve.

Exercise 1. A public company has in year k profits amounting to x_k SEK. The management then distributes u_k to the shareholders and invests $x_k - u_k$ in the company itself. Each SEK invested in such way will increase the company profit by $\theta > 0$ the following year so that

$$x_{k+1} = x_k + \theta(x_k - u_k).$$

Suppose $x_0 \geq 0$ and $0 \leq u_k \leq x_k$ so that $x_k \geq 0$ for each k. The objective of the management is to maximize the total amount distributed to the shareholders over a period of N years, i.e.,

$$\max_{u_k} \sum_{k=0}^{N-1} u_k$$

subject to $u_k \in [0, x_k]$.

Exercise 2. Derive the complete value iteration procedure – with $J_0 = 0$ – for the optimal control problem

$$\min \sum_{i=1}^{\infty} (x_k^\top Q x_k + u_k^\top R u_k)$$

under the constraint:

$$x_{k+1} = Ax_k + Bu_k.$$

(there is no constraint on u).

Hint: 1) Write J_k, $k \geq 1$ as $J_k(x) = x^\top P_k x$, and $u_k = -K_k x$ where

$$u_k \in \arg\min_u \{x^\top Q x + u^\top R u + J_k(Ax + Bu)\}.$$

2) find the iteration formula for K_k and P_k. Don't forget the boundary conditions.

Exercise 3. Show that free terminal time optimal control problem can be turned into a fixed terminal time problem. Why is this useful in numerical computation? *Hint: consider a rescaling of time $\tau = \frac{t}{t_f}$.*

Exercise 4. Derive the maximum principle for the Bolza form cost by utilizing the maximum principle for the Mayer form.

Exercise 5. Prove the maximum principle for the case that t_f is free. You may consider the Mayer type problem. *Hint: all the necessary conditions for t_f fixed are still necessary. One only needs to derive the additional condition that $H \equiv 0$ along the optimal solution. You can either use the trick in Ex2 or consider a new variation in t_f: $x_\epsilon(t_f + \epsilon\mu) \in \Omega_1$ where $x_\epsilon(\cdot)$ is some needle variation and μ some real number.*