# Exercise for Optimal control – Week 3

Choose **1.5** problems to solve.

**Exercise 1.** Consider a harmonic oscillator $\ddot{x} + x = u$ whose control is constrained in the interval $[-1, 1]$. Find an optimal controller $u$ which drives the system at initial state $(x(0), \dot{x}(0)) = (X_1, X_2)$ to the origin in minimal time. Draw the phase plot.

**Exercise 2.** Consider a rocket, modeled as a particle of constant mass $m$ moving in zero gravity empty space. Let $u > 0$ be the mass flow, assumed to be a known function of time, let $c$ be the constant thrust velocity and $v$ an angle that can be controlled. See Figure 1. The equations of motion are

$$\begin{aligned}\dot{x}_1 &= x_3\\ \dot{x}_2 &= x_4\\ \dot{x}_3 &= \frac{c}{m}u(t)\cos(v(t))\\ \dot{x}_4 &= \frac{c}{m}u(t)\sin(v(t))\end{aligned}$$

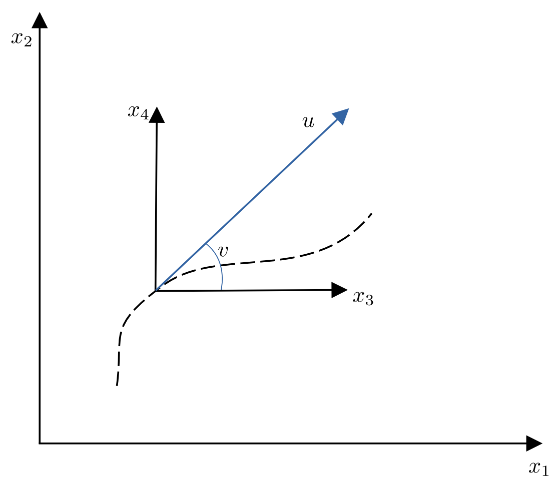


Figure 1: A rocket model.

1) Show that cost functionals of the class

$$\min_{v(\cdot)} \int_0^{t_f} \mathrm{d}t \text{ or } \min_{v(\cdot)} \phi(x(t_f))$$

gives the optimal control

$$\tan v^*(t) = \frac{c_1 + c_2 t}{c_3 + c_4 t}.$$

2) Assume that the rocket starts at rest at the origin and that we want to drive it to a given height $x_{2f}$ in a given time $t_f$ such that the final velocity in the horizontal direction $x_3(t_f)$ is maximized while $x_{4f} = 0$. Show that the optimal control is reduced to a linear tangent law

$$\tan v^*(t) = c_1 + c_2 t.$$

3) Let the rocket in represent a missile whose target is at rest. Minimize the transfer time $t_f$from the state $[0, 0, x_{3i}, x_{4i}]$ to the state $[x_{1f}, x_{2f}, \text{free}, \text{free}]$. Solve the problem under the assumption that $u$ is constant.

4) To increase the realism now assume that the motion is under a constant gravitational force. The only equation that needs to be modified is the one for $x_4$ (the acceleration in the vertical direction):

$$\dot{x}_4 = \frac{c}{m} u(t) \sin(v(t)) - g.$$

Show that the optimal law is still optimal for the cost functional

$$\min_{v(\cdot)} \phi(x(t_f)) + \int_0^{t_f} \mathrm{d}t.$$

5) Now we take into consideration of the mass loss of the rocket. Let $x_5$ denote the mass of the rocket. The overall equations of motion now read

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \frac{c}{m} u(t) \cos(v(t)) \\
\dot{x}_4 &= \frac{c}{m} u(t) \sin(v(t)) - g \\
\dot{x}_5 &= -u(t)
\end{aligned}$$

where $u \in [0, u_{\max}]$. Show that the optimal solution to transferring the rocket from a state of given position, velocity and mass to a given altitude $x_{2f}$ using a given amount of fuel, such that the distance $x_1(t_f) - x_1(0)$ is maximized, is

$$v^*(t) = \text{constant}, \;\; u^*(t) = \{u_{\max}, 0\}.$$

**Exercise 3.** Try to solve the Rayleigh problem: consider minimizing

$$J = \int_0^{t_f} (u^2 + x_1^2) \mathrm{d}t$$

subject to (the controlled van de Pol oscillator):

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + x_2(1.4 - 0.14 x_2^2) + 4u
\end{aligned}$$

with initial condition $(x_1(0), x_2(0)) = (-5, -5)$, $t_f = 4.5$ and a mixed input and state constraint:

$$-1 \leq u(t) + \frac{x_1(t)}{6} \leq 0.$$

Draw the optimal controller and the state trajectory. You may use numerical methods, e.g., discretization.