Exercise for Optimal control – Week 1

Choose two problems to solve.

Exercise 1 (Fundamental lemma of CoV)**.** Let f be a real valued function defined on open interval (a,b) and f satisfies

$$\int_a^b f(x)h(x)\mathrm{d}x = 0$$

for all $h \in C_c(a,b)$, i.e., h is continuous on (a,b) and its support, i.e., the closure of

$$\{x : h(x) \neq 0\}$$

is contained in (a,b).

1) Show that f is identically zero if f is continuous. If f is only piecewise continuous, then f has only finite non-zero values. *Hint: consider for example that $f(x_1) > 0$, since f is continuous, f should be positive near x_1, say $(x_1 - \delta, x_1 + \delta)$. Next, construct a non-negative continuous function $h \in C_c(x_1 - \epsilon, x_1 + \epsilon)$ which has some positive values on this interval. You'll then arrive at a contradiction since $\int_a^b fh\mathrm{d}x > 0$.*

2) Extend to multivariate case: i.e., if f is continuous on an open set Ω and

$$\int_\Omega f(x)h(x)\mathrm{d}x = 0$$

for all $h \in C_c(\Omega)$, then $f \equiv 0$ for all $x \in \Omega$. *Hint: as before, if $f(x_1) > 0$ at some x_1, then there exists a neighborhood of x_1 on which f is positive. Construct a non-negative function which vanishes outside this neighborhood. And you get a contradiction.*

Exercise 2 (Naive derivation of the 1st variations)**.** Derive the first order variations of the optimal control problem

$$\begin{aligned}
&\dot{x} = f(x,u) \\
&\min_u J(u(\cdot)) := \varphi(x(T)) + \int_0^T L(x,u)\mathrm{d}t
\end{aligned}$$

using variation/perturbation

$$u_\epsilon(t) = u_*(t) + \epsilon v(t)$$

for an arbitrary $v(\cdot)$. Assume f, L and φ are C^2 and the initial state is fixed. Compare with the computation using δ operator. *Hint: write $x_\epsilon(t)$ as the solution to the following integral equation $\dot{x}_\epsilon(t) = f(x_\epsilon, u_\epsilon)$.* Define $H(x,u,p) = p^\top f(x,u) - L(x,u)$ (the Hamiltonian!), and fix a C^1 curve $t \mapsto p(t)$, then

$$\begin{aligned}
J(u_\epsilon) &= \varphi(x_\epsilon(T)) + \int_0^T L(x_\epsilon, u_\epsilon)\mathrm{d}t \\
&= \varphi(x_\epsilon(T)) + \int_0^T p^\top(t)\dot{x}_\epsilon - H(x_\epsilon, u_\epsilon, p)\mathrm{d}t
\end{aligned}$$

Apply integration by parts to get rid of $\dot{x}_\epsilon$. Then compute $\frac{\partial J}{\partial \epsilon}|_{\epsilon=0}$.

Exercise 3 (Dido's problem)**.** Formulate Dido's problem as optimal control with only finite dimensional constraints. *Hint: define a new variable $\eta(t) := \int_0^t |\gamma'(s)|\mathrm{d}s$ and let $\gamma'(t) := u(t)$.*

Exercise 4 (Minimum surface problem)**.** Let D be an open set on the plane $\mathbb{R}^2$. Consider a surface $D \ni (x,y) \mapsto (x,y,u(x,y)) \in \mathbb{R}^3$. The area of the graph of this surface can be calculated as

$$\mathcal{A}(u) = \int_D \sqrt{1+u_x^2+u_y^2}\mathrm{d}x\mathrm{d}y$$

where u_x, u_y stands for partial derivatives, with boundary condition $u|_{\partial D} = g$ for some function g on ∂D. Consider the problem of minimizing $\mathcal{A}(\cdot)$. Derive the Euler-Lagragian equation of this problem. *Hint: let v be an arbitrary function whose support lies in D. Assume u_* is a minimizer, and consider the variation*

$$u_\epsilon = u + \epsilon v.$$

Calculate $\frac{\partial \mathcal{A}(u_\epsilon)}{\partial \epsilon}|_{\epsilon=0}$ *to get*

$$\int_D \frac{u_x v_x + u_y v_y}{\sqrt{1+u_x^2+u_y^2}}\mathrm{d}x\mathrm{d}y = 0$$

They apply integration by parts formula for $\varphi \in C_c^\infty(D)$ (no boundary terms)

$$\int_D v_x \varphi \mathrm{d}x\mathrm{d}y = -\int_D v\varphi_x \mathrm{d}x\mathrm{d}y.$$

After eliminating v_x, v_y, apply Exercise 1.

Exercise 5 (Boundary value problem)**.** Consider the following boundary value problem

$$\ddot{q} + f(q,\dot{q}) = 0 \tag{1}$$

where $q \in \mathbb{R}^n$ with boundary condition

$$q(0) = a, \quad q(1) = b. \tag{2}$$

Two classical methods exist for numerically solving the BVP: 1) Finite difference method; 2) Shooting.

1) The first order derivative may be discretized as

$$\dot{q}(t_i) \approx \frac{q(t_{i+1}) - q(t_i)}{t_{i+1} - t_i}$$

In particular, for fixed step size $h = t_{i+1} - t_i$,

$$\dot{q}_i = \frac{q_{i+1} - q_i}{h}.$$

Propose a discretization scheme for second order derivative and write the discretized version of (1), (2) in matrix form. Use this to find the geodesic on the ellipsoid

$$x^2 + y^2 + \frac{z^2}{4} = 1.$$

Try different boundary values and observe the non-uniqueness geodesic phenomenon.

2) The shooting method works like this. Write the second order ODE as a first order one by defining $x_1 = q$, $x_2 = \dot{q}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -f(x_1,x_2) \end{bmatrix}$$

Then we know $x_1(0) = a$, $x_1(1) = b$. Suppose that $x_2(0) = \lambda$. Then with the initial condition $x(0) = (a, \lambda)$, the value $x_1(1)$ should be uniquely determined, which is a function of λ, denoted as $x_1(1,\lambda)$. Let

$$F(\lambda) := x_1(1,\lambda) - b.$$

The idea is to alter λ so that $F(\lambda)$ converges to 0. This is equivalent to finding the root of F (or $|F(\cdot)|^2$). Solve the problem in 1) using shooting method. You may use gradient descent or Newton's method. Is the scheme stable? Or, when does the algorithm converge?

Exercise 6 (Hamiltonian equation)**.** Recall the Hamiltonian equation:

$$\dot{q} = \frac{\partial H}{\partial p}(q, p)$$
$$\dot{p} = -\frac{\partial H}{\partial q}(q, p)$$

where $q, p \in \mathbb{R}^n$. Let ϕ_t be the flow of the system. That is, $(q(t), p(t)) = \phi_t(q_0, p_0)$ for the initial condition (q_0, p_0).

1) For functions $H_1, H_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, define the Poisson bracket between H_1 and H_2 as

$$\{H_1, H_2\} = \sum_{i=1}^{n} \frac{\partial H_1}{\partial q_i} \frac{\partial H_2}{\partial p_i} - \frac{\partial H_1}{\partial p_i} \frac{\partial H_2}{\partial q_i}.$$

Show that for any real function f on $\mathbb{R}^n \times \mathbb{R}^n$, f satisfies the ordinary differential equation along the Hamiltonian system:

$$\dot{f} = \{f, H\}.$$

2) Given a bounded set $U \subseteq \mathbb{R}^n \times \mathbb{R}^n$, define

$$U_t = \phi_t(U)$$

Show that the volume of U_t is constant. *Hint: use the transport equation: consider a system $\dot{x} = f(x)$, and let $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ be its flow, then for any bounded set $D \subseteq \mathbb{R}^n$,*

$$\frac{d}{dt}\text{vol}(\phi_t(D)) = \int_{\phi_t(D)} \text{div} f \,\text{d}x.$$

3) Assume that there exists a bounded forward invariant set $D \subseteq \mathbb{R}^n \times \mathbb{R}^n$ of the Hamiltonian system. Then for any open set $U \subseteq D$, and any $s > 0$, there exists at least one point $x \in U$ which returns to U after some time $t \geq s$.

4) The Hamiltonian equation has time-dependent version

$$\dot{q} = \frac{\partial H}{\partial p}(q, p, t)$$
$$\dot{p} = -\frac{\partial H}{\partial q}(q, p, t)$$

Show that the energy is not preserved.

5) The Hamiltonian equation can be generalized to

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & R(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G(q)u \end{bmatrix}$$

where u is an input and R, G are matrices. Assume that $H \geq 0$. Show that the matrix $R(q)$ plays the role of energy damping/injection – depending on sign. Show that the system is dissipative if R is semi-positive definite, in the sense that there exists an output y such that

$$-\int_{\infty}^{\infty} y^\top(t) u(t) \text{d}t \leq H(q(0), p(0)).$$

That is, the energy that can be extracted from the system via the input output pair (u, y) is less than the total energy of the system.