Exercise for Optimal control – Week 1

Choose **two** problems to solve.

Disclaimer

This is not a complete solution manual. For some of the exercises, we provide only partial answers, especially those involving numerical problems. If one is willing to use the solution manual, one should judge whether the solutions are correct or wrong him/herself.

Exercise 1 (Fundamental lemma of CoV)**.** Let f be a real valued function defined on open interval (a, b) and f satisfies

$$\int_a^b f(x)h(x)\mathrm{d}x = 0$$

for all $h \in C_c(a, b)$, i.e., h is continuous on (a, b) and its support, i.e., the closure of

$$\{x : h(x) \neq 0\}$$

is contained in (a, b). Note that this also holds for $h \in C_c^1(a, b)$ or $h \in C_c^\infty(a, b)$.

1) Show that f is identically zero if f is continuous. If f is only piecewise continuous, then f has only finite non-zero values.

2) Extend to multivariate case: i.e., if f is continuous on an open set Ω and

$$\int_\Omega f(x)h(x)\mathrm{d}x = 0$$

for all $h \in C_c(\Omega)$, then $f \equiv 0$ for all $x \in \Omega$.

Solution. 1) Suppose that f is positive at x_0. Since f is continuous, then there exists an interval, say $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$ on which f is positive. Define

$$h(x) = \begin{cases} \frac{\delta^2}{4} - (x - x_0)^2, & x_0 - \frac{\delta}{2} \leq x \leq x_0 + \frac{\delta}{2} \\ 0, & \text{else} \end{cases}.$$

then $h \in C_c(a, b)$ – h is even $C_c^1(a, b)$. But

$$\int_a^b h(x)f(x)\mathrm{d}x = \int_{x_0-\delta/2}^{x_0+\delta/2} h(x)f(x)\mathrm{d}x > 0$$

a contradiction. When f is only piecewise continuous, we can reason as in the continuous case on continuous intervals.

2) If f is positive at some point $x_0 \in D$, then there exists a ball centered at x_0 with radius r, denoted $B(x_0, r)$. Construct a function

$$h(x) = \begin{cases} \frac{r^2}{4} - |x - x_0|^2, & |x - x_0| \leq \frac{r}{2} \\ 0, & \text{else} \end{cases}.$$

then h is continuous but $\int fh > 0$, again a contradiction.

Exercise 2 (Naive derivation of the 1st variations)**.** Derive the first order necessary condition of the optimal control problem

$$\dot{x} = f(x, u)$$

$$\min_u J(u(\cdot)) := \varphi(x(T)) + \int_0^T L(x, u)\mathrm{d}t$$

using variation/perturbation

$$u_\epsilon(t) = u_*(t) + \epsilon v(t)$$

for an arbitrary $v(\cdot)$. Assume f, L and φ are C^1 and the initial state is fixed. Compare with the computation using δ operator.

Solution. Define $H(x,u,p) = p^\top f(x,u) - L(x,u)$ (the Hamiltonian), and fix a C^1 curve $t \mapsto p(t)$, then

$$\begin{aligned} J(u_\epsilon) &= \varphi(x_\epsilon(T)) + \int_0^T L(x_\epsilon, u_\epsilon)\mathrm{d}t \\ &= \varphi(x_\epsilon(T)) + \int_0^T p^\top(t)\dot{x}_\epsilon - H(x_\epsilon, u_\epsilon, p)\mathrm{d}t \end{aligned}$$

where x_ϵ is the solution to

$$\dot{x}_\epsilon = f(x_\epsilon, u_\epsilon), \quad x_\epsilon(0) = x_0.$$

Applying integration by parts we can get rid of $\dot{x}_\epsilon$:

$$J(u_\epsilon) = \varphi(x_\epsilon(T)) + p(t)^\top x_\epsilon(t)|_0^T - \int_0^T \dot{p}^\top(t)x_\epsilon(t) + H(x_\epsilon, u_\epsilon, p)\mathrm{d}t$$

Now calculate

$$\begin{aligned} \frac{\partial J}{\partial \epsilon}|_{\epsilon=0} &= \varphi_x(x_*(T))\frac{\partial x_\epsilon}{\partial \epsilon}|_{\epsilon=0} + p(t)^\top \frac{\partial x_\epsilon}{\partial \epsilon}|_{\epsilon=0} - \int_0^T \dot{p}^\top(t)\frac{\partial x_\epsilon}{\partial \epsilon}|_{\epsilon=0} + H_x\frac{\partial x_\epsilon}{\partial \epsilon}|_{\epsilon=0} + H_u v\mathrm{d}t \\ &= (\varphi_x(x_*(T)) + p(T)^\top)\frac{\partial x_\epsilon}{\partial \epsilon}|_{\epsilon=0} - \int_0^T (\dot{p}^\top + H_x)\frac{\partial x_\epsilon}{\partial \epsilon}|_{\epsilon=0} + H_u v\mathrm{d}t. \end{aligned}$$

Choose p such that

$$\dot{p} = -H_x^\top(x_*, u_*, p)$$

with terminal condition

$$p(T) = -\varphi_x(x_*(T))^\top$$

we arrive at

$$\frac{\partial J}{\partial \epsilon}|_{\epsilon=0} = -\int_0^T H_u v\mathrm{d}t.$$

Since v is arbitrary, we get by the fundamental Lemma of CoV that $H_u(x_*(t), u_*(t), p(t)) = 0$ for all $t \in [0,T]$. This is consistent with the calculation using δ operator.

Exercise 3 (Dido's problem)**.** Formulate Dido's problem as optimal control with only finite dimensional constraints.

Solution. In Dido's problem, the area to be maximize is some functional

$$J(\gamma) = \int_0^1 L(\gamma, \dot{\gamma})\mathrm{d}s$$

where $\gamma = [\gamma_1, \gamma_2]^\top$ is some curve on the xy-plane with the constraints $\gamma_2(0) = \gamma_2(1) = 0$, $\gamma_2(s) \geq 0$ for all $s \in [0,1]$. The constraint for this problem is

$$\int_0^1 |\gamma'(s)|\mathrm{d}s = C$$

for some constant $C > 0$. Introduce a new state

$$\eta(t) = \int_0^t |\gamma'(s)|\mathrm{d}s$$

and input $u(t) = \gamma'(t)$. Then the system equation may be written as

$$\dot{\gamma} = u$$
$$\dot{\eta} = |u|$$

and the cost

$$J(u) = \int_0^1 L(\gamma, u)\mathrm{d}t$$

with constraint

$$\gamma_2(0) = \gamma_2(1) = 0,\ \gamma_2 \geq 0,$$
$$\eta(0) = 0,\ \eta(1) = C$$

Exercise 4 (Minimum surface problem)**.** Let D be an open set on the plane $\mathbb{R}^2$. Consider a surface $D \ni (x, y) \mapsto (x, y, u(x, y)) \in \mathbb{R}^3$. The area of the graph of this surface can be calculated as

$$\mathcal{A}(u) = \int_D \sqrt{1 + u_x^2 + u_y^2}\mathrm{d}x\mathrm{d}y$$

where u_x, u_y stands for partial derivatives, with boundary condition $u|_{\partial D} = g$ for some function g on ∂D. Consider the problem of minimizing $\mathcal{A}(\cdot)$. Derive the Euler-Lagragian equation of this problem.

Solution. Let v be an arbitrary function whose support lies in D. Assume u_* is a minimizer, and consider the variation

$$u_\epsilon = u + \epsilon v.$$

Calculate $\frac{\partial \mathcal{A}(u_\epsilon)}{\partial \epsilon}|_{\epsilon=0}$ to get

$$\int_D \frac{u_x v_x + u_y v_y}{\sqrt{1 + u_x^2 + u_y^2}}\mathrm{d}x\mathrm{d}y = 0$$

After integration by parts, we get

$$\int_D \frac{d}{dx}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right)v + \frac{d}{dy}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right)v = 0.$$

Since $v \in C_c^1(D)$ is arbitrary, by the fundamental lemma of CoV, we get

$$\frac{d}{dx}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right)v + \frac{d}{dy}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0.$$

Exercise 5 (Boundary value problem)**.** Consider the following boundary value problem

$$\ddot{q} + f(q, \dot{q}) = 0 \tag{1}$$

where $q \in \mathbb{R}^n$ with boundary condition

$$q(0) = a, \quad q(1) = b. \tag{2}$$

Two classical methods exist for numerically solving the BVP: 1) Finite difference method; 2) Shooting.

1) The first order derivative may be discretized as

$$\dot{q}(t_i) \approx \frac{q(t_{i+1}) - q(t_i)}{t_{i+1} - t_i}$$

In particular, for fixed step size $h = t_{i+1} - t_i$,

$$\dot{q}_i = \frac{q_{i+1} - q_i}{h}.$$

Propose a discretization scheme for second order derivative and write the discretized version of (1), (2) in matrix form. Use this to find the geodesic on the ellipsoid

$$x^2 + y^2 + \frac{z^2}{4} = 1.$$

Try different boundary values and observe the non-uniqueness geodesic phenomenon.

2) The shooting method works like this. Write the second order ODE as a first order one by defining $x_1 = q$, $x_2 = \dot{q}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -f(x_1, x_2) \end{bmatrix}$$

Then we know $x_1(0) = a$, $x_1(1) = b$. Suppose that $x_2(0) = \lambda$. Then with the initial condition $x(0) = (a, \lambda)$, the value $x_1(1)$ should be uniquely determined, which is a function of λ, denoted as $x_1(1, \lambda)$. Let

$$F(\lambda) := x_1(1, \lambda) - b.$$

The idea is to alter λ so that $F(\lambda)$ converges to 0. This is equivalent to finding the root of F (or $|F(\cdot)|^2$). Solve the problem in 1) using shooting method. You may use gradient descent or Newton's method. Is the scheme stable? Or, when does the algorithm converge?

Solution. 1) Let $t_i = ih$, $i = 0, \cdots, N$ and $h = 1/N$. For second order derivative, we can use

$$\ddot{q}(t_i) = \frac{\dot{q}(t_{i+1}) - \dot{q}(t_i)}{t_{i+1} - t_i} = \frac{\frac{q(t_{i+1}) - q(t_i)}{t_{i+1} - t_i} - \frac{q(t_i) - q(t_{i-1})}{t_i - t_{i-1}}}{t_{i+1} - t_i}$$

whenever $1 \leq i \leq N - 1$. Since $t_{i+1} - t_i = h$, the above can be simplified to

$$\ddot{q}(t_i) = \frac{q(t_{i+1}) - 2q(t_i) + q(t_{i-1})}{h^2}$$

Denote $q(t_i) = q_i$ and $q = [q_1, \cdots, q_{N-1}]^\top$, then $\ddot{q} = [\ddot{q}_1, \cdots, \ddot{q}_{N-1}]^\top = \frac{1}{h^2}(Aq + d)$ where

$$A = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & 2 \end{bmatrix}, \quad d = \begin{bmatrix} q(0) \\ 0 \\ \vdots \\ q(1) \end{bmatrix}.$$

On the other hand, $\dot{q}$ can also be expressed as a function of $[q_1, \cdots, q_{N-1}]^\top$. Thus the discretization for $\ddot{q} + f(q, \dot{q}) = 0$ can be written in the form

$$Aq + h^2 f(q) + d = 0.$$

This a nonlinear equation in q, which can be solved using e.g., Newton's method. Alternatively, one can use the so called shooting method.

Let us derive the geodesic equation on the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$. A point on the ellipsoid can be described by two angles: draw the segment connecting the point and the origin, whose length is d. Let ϕ be the angle between this segment and the z axis. Then project the point to the xy-plane and connect this point with the origin using another segment. The angle between this segment and the x-axis is denoted θ. The the point is

$$\begin{aligned} x &= d \sin\phi \cos\theta \\ y &= d \sin\phi \sin\theta \\ z &= d \cos\phi. \end{aligned}$$

Since $x^2 + y^2 + \frac{z^2}{4} = 1$, we can find

$$d^2(\sin^2\phi + \frac{\cos^2\phi}{4}) = 1$$

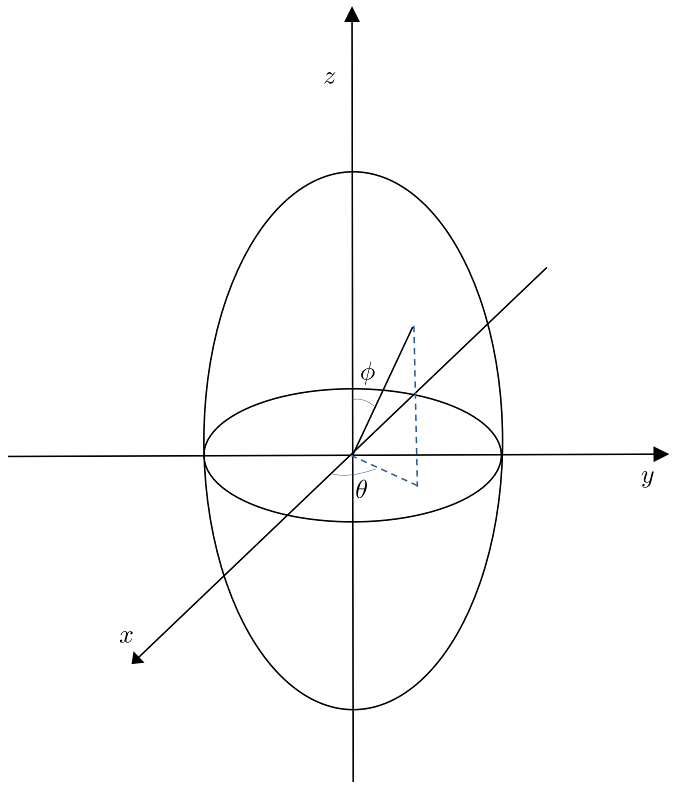

Figure 1: Ellipsoid.

or

$$d = \frac{1}{\sqrt{1 + 3\sin^2\phi}}.$$

Thus

$$x = \frac{\sin\phi\cos\theta}{\sqrt{1 + 3\sin^2\phi}}$$
$$y = \frac{\sin\phi\sin\theta}{\sqrt{1 + 3\sin^2\phi}}$$
$$z = \frac{\cos\phi}{\sqrt{1 + 3\sin^2\phi}}$$

Let $w_\theta = \frac{\partial w}{\partial \theta}$ and $w_\phi = \frac{\partial w}{\partial \phi}$ where w can be x, y, z. Then

Given a curve on the ellipsoid described by $t \mapsto (\phi(t), \theta(t))$, we can calculate the length of the curve:

$$\int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\mathrm{d}t = \int_0^1 \sqrt{(x_\theta^2 + y_\theta^2 + z_\theta^2)\dot{\theta}^2 + (x_\phi^2 + y_\phi^2 + z_\phi^2)\dot{\phi}^2}\mathrm{d}t.$$

To get the geodesic between two fixed points, it is equivalent to minimize

$$J(\theta(\cdot), \phi(\cdot)) := \int_0^1 (x_\theta^2 + y_\theta^2 + z_\theta^2)\dot{\theta}^2 + (x_\phi^2 + y_\phi^2 + z_\phi^2)\dot{\phi}^2\mathrm{d}t.$$

Now

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= 2(x_\theta x_{\theta\theta} + y_\theta y_{\theta\theta} + z_\theta z_{\theta\theta})\dot{\theta}^2 + 2(x_\theta x_{\phi\theta} + y_\phi y_{\phi\theta} + z_\phi z_{\phi\theta})\dot{\phi}^2 \\
\frac{\partial L}{\partial \phi} &= 2(x_\theta x_{\theta\phi} + y_\theta y_{\theta\phi} + z_\theta z_{\theta\phi})\dot{\theta}^2 + 2(x_\phi x_{\phi\phi} + y_\phi y_{\phi\phi} + z_\phi z_{\phi\phi})\dot{\phi}^2 \\
\frac{\partial L}{\partial \dot{\theta}} &= 2(x_\theta^2 + y_\theta^2 + z_\theta^2)\dot{\theta} \\
\frac{\partial L}{\partial \dot{\phi}} &= 2(x_\phi^2 + y_\phi^2 + z_\phi^2)\dot{\phi} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 2(2x_\theta x_{\theta\theta}\dot{\theta} + 2x_\theta x_{\theta\phi}\dot{\phi} + 2y_\theta y_{\theta\theta}\dot{\theta} + 2y_\theta y_{\theta\phi}\dot{\phi} + 2z_\theta z_{\theta\theta}\dot{\theta} + 2z_\theta z_{\theta\phi}\dot{\phi})\dot{\theta} + 2(x_\theta^2 + y_\theta^2 + z_\theta^2)\ddot{\theta} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= 2(2x_\phi x_{\phi\theta}\dot{\theta} + 2x_\phi x_{\phi\phi}\dot{\phi} + 2y_\phi y_{\phi\theta}\dot{\theta} + 2y_\phi y_{\phi\phi}\dot{\phi} + 2z_\phi z_{\phi\theta}\dot{\theta} + 2z_\phi z_{\phi\phi}\dot{\phi})\dot{\theta} + 2(x_\phi^2 + y_\phi^2 + z_\phi^2)\ddot{\phi}
\end{aligned}$$

Write the EL equation as a boundary value problem:

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + f\left(\begin{bmatrix} \theta \\ \phi \end{bmatrix}, \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} \right) = 0$$

with fixed boundary conditions. Then after discretization, we may use Newton method to solve the boundary value problem.

2) To apply the shooting method, we assume $(\dot{\theta}(0), \dot{\phi}(0)) = (b_1, b_2)$ for some initial value and then use ode45/23 to solve the initial value problem to get $(\theta(1), \phi(1))^\top$ – depending on (b_1, b_2). Note that $(\theta(1), \phi(1))$ is a function of (b_1, b_2), say $(\theta(1), \phi(1))^\top = F(b_1, b_2)$. Define $G(b_1, b_2) := ||F(b_1, b_2) - (\theta(1), \phi(1))^\top||^2$, then it suffices to find the root of G. The apply for example gradient descent method to solve the nonlinear equation:

$$\begin{aligned}
b_1^{(k+1)} &= b_1^{(k)} - \gamma_1 \frac{G(b_1^{(k)}, b_2^{(k)}) - G(b_1^{(k-1)}, b_2^{(k)})}{b_1^{(k)} - b_1^{(k-1)}} \\
b_2^{(k+1)} &= b_2^{(k)} - \gamma_2 \frac{G(b_1^{(k)}, b_2^{(k)}) - G(b_1^{(k)}, b_2^{(k-1)})}{b_2^{(k)} - b_2^{(k-1)}}
\end{aligned}$$

Exercise 6 (Hamiltonian equation)**.** Recall the Hamiltonian equation:

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q, p) \\
\dot{p} &= -\frac{\partial H}{\partial q}(q, p)
\end{aligned}$$

where $q, p \in \mathbb{R}^n$. Let ϕ_t be the flow of the system. That is, $(q(t), p(t)) = \phi_t(q_0, p_0)$ for the initial condition (q_0, p_0).

1) For functions $H_1, H_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, define the Poisson bracket between H_1 and H_2 as

$$\{H_1, H_2\} = \sum_{i=1}^n \frac{\partial H_1}{\partial q_i}\frac{\partial H_2}{\partial p_i} - \frac{\partial H_1}{\partial p_i}\frac{\partial H_2}{\partial q_i}.$$

Show that for any real function f on $\mathbb{R}^n \times \mathbb{R}^n$, f satisfies the ordinary differential equation along the Hamiltonian system:

$$\dot{f} = \{f, H\}.$$

2) Given a bounded set $U \subseteq \mathbb{R}^n \times \mathbb{R}^n$, define

$$U_t = \phi_t(U)$$

Show that the volume of U_t is constant. *Hint: use the transport equation: consider a system $\dot{x} = f(x)$, and let $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ be its flow, then for any bounded set $D \subseteq \mathbb{R}^n$,*

$$\frac{d}{dt}\mathrm{vol}(\phi_t(D)) = \int_{\phi_t(D)} \mathrm{div} f \mathrm{d}x.$$

3) Assume that there exists a bounded forward invariant set $D \subseteq \mathbb{R}^n \times \mathbb{R}^n$ of the Hamiltonian system. Then for any open set $U \subseteq D$, and any $s > 0$, there exists at least one point $x \in U$ which returns to U after some time $t \geq s$.

4) The Hamiltonian equation has time-dependent version

$$\dot{q} = \frac{\partial H}{\partial p}(q, p, t)$$
$$\dot{p} = -\frac{\partial H}{\partial q}(q, p, t)$$

Show that the energy is not preserved.

5) The Hamiltonian equation can be generalized to

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & -R(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G(q)u \end{bmatrix}$$

where u is an input and R, G are matrices. Assume that $H \geq 0$. Show that the matrix $R(q)$ plays the role of energy damping/injection – depending on sign. Show that the system is dissipative if R is semi-positive definite, in the sense that there exists an output y such that

$$-\int_{\infty}^{\infty} y^\top(t)u(t)\mathrm{d}t \leq H(q(0), p(0)).$$

That is, the energy that can be extracted from the system via the input output pair (u, y) is less than the total energy of the system.

Solution. 1) By definition,

$$\begin{aligned} \dot{f} &= \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial f}{\partial p_i}\dot{p}_i \\ &= \sum_i \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \\ &= \{f, H\}. \end{aligned}$$

2) Differentiate $\mathrm{vol}(\phi_t(D))$ and apply the transport equation:

$$\frac{d}{dt}\mathrm{vol}(\phi_t(D)) = \int_{\phi_t(D)} \mathrm{div}(H_p, -H_q)\mathrm{d}(q, p) = \int_{\phi_t(D)} \sum_i \frac{\partial^2 H}{\partial p_i \partial q_i} - \frac{\partial^2 H}{\partial q_i \partial p_i}\mathrm{d}(q, p) = 0$$

3) Define

$$U_j := \phi_{js}(U), \quad j = 0, \cdots, m, \cdots$$

Since D is invariant, $U_j \subset D$ for all $j \geq 0$. Since Hamiltonian system is volume preserving, and that D is bounded – having finite volume – then there must exists j, k, say $k > j$ such that

$$U_j \cap U_k \neq \emptyset$$

otherwise the total volume of $U_0, U_j \cdots$ is infinite, a contradiction. Therefore,

$$\phi_{js}(U) \cap \phi_{ks}(U) \neq \emptyset.$$

Let $x \in \phi_{js}(U) \cap \phi_{ks}(U)$, then there exist $y_1, y_2 \in U$ such that $x = \phi_{js}(y_1) = \phi_{ks}(y_2)$. Hence $y_1 = \phi_{(k-j)s}(y_2)$, which says that y_2 returns to U after $(k - j)s$ time.

4) Straightforward: $\frac{dH}{dt} = \frac{\partial H}{\partial t} \neq 0$.

5) Notice

$$\begin{aligned} \frac{dH}{dt} &= H_q\dot{q} + H_p\dot{p} = H_q H_p^\top - H_p(H_q^\top - R(q)H_p^\top + G(q)u) \\ &= -H_p R(q) H_p^\top + H_p G(q)u. \end{aligned}$$

If $u = 0$, we have

$$\frac{dH}{dt} = -H_p R(q) H_p^\top$$

In particular, if $R(q) > 0$, the energy decreases and when $R(q) < 0$, the energy of the system increases. Now assume $R(q) > 0$, let $y = G(q)^\top H_p^\top$, and integrate the inequality

$$\frac{dH}{dt} \leq y^\top u,$$

we immediately get

$$H(x(t)) \leq H(x(0)) + \int_0^\top y^\top u \mathrm{d}t$$

or

$$-\int_0^T y^\top u \mathrm{d}t \leq H(x(0)) - H(x(t)) \leq H(x(0))$$

where we used $H \geq 0$. Now since t is arbitrary, the conclusion immediately follows.