

## Performance specifications

Driving example: Cruise control Design trade-offs Expression in the frequency-domain Loopshaping

#### Fundamental limitations

System design considerations Sensitivity minimization Bode's integral formula Gain crossover frequency inequality Internal stability





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#### **Requirements:**

- Stability
- Reference tracking
- Disturbance rejection
- Noise attenuation
- Robustness to process variations

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#### **Requirements:**

- Stability
- Reference tracking
- Disturbance rejection
- Noise attenuation
- Robustness to process variations

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#### The Gang of Four:

$$S = \frac{1}{1+PC}$$

$$T = \frac{PC}{1+PC}$$

$$CS = \frac{C}{1+PC}$$

$$RC = \frac{P}{1+PC}$$

$$\bullet PS = \frac{P}{1+PC}$$

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## Nyquist plot

- open-loop L = PC
- stability criteria
- margins





Two tools

#### **Bode plot**

- closed-loop
- performances of the Gang of Four

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- 2 Fundamentals: problem formulation
  - System representation and feedback basics
  - Specifications and performance limitations
- 3 Design techniques

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 $\dot{mv} = -bv - mgsin( heta) + u$  $sin( heta) \sim heta$ 



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## Follow a reference signal: the servo problem

• Act on  $T = \frac{V(s)}{R(s)} = \frac{PC}{1+PC}$ 

In time-domain, we want:

- Rise time < 5s
- Overshoot < 10%</p>
- Steady-state error < 2%



Figure: Step response of the system to u = 500N.

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## Follow a reference signal: the servo problem

Act on  $T = \frac{V(s)}{R(s)} = \frac{PC}{1+PC}$ 





## Figure: Step responses of the closed-loop for different $k_{p}$ .

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Effect on noise sensitivity: 
$$T_{n \rightarrow u} = -\frac{C}{1+PC}$$



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Design trade-offs

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Effect on noise sensitivity: T<sub>n→u</sub> = -C/(1+PC)
Effect on the command: T<sub>r→u</sub> = C/(1+PC) → Limitations of the actuators!



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Design trade-offs

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Effect on noise sensitivity: T<sub>n→u</sub> = - C/(1+PC)
Effect on the command: T<sub>r→u</sub> = C/(1+PC) → Limitations of the actuators!
Effect on disturbance rejection T<sub>r→v</sub> = P/(1+PC)



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Design trade-offs

- Effect on noise sensitivity:  $T_{n \rightarrow u} = -\frac{C}{1+PC}$
- Effect on the command:  $T_{r \rightarrow u} = \frac{c}{1 + PC} \rightarrow$  Limitations of the actuators!
- Effect on disturbance rejection  $T_{r \rightarrow v} = \frac{P}{1 + PC}$
- Effect on robustness



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**Stability and robustness**: gain margin  $g_m$ , phase margin  $\phi_m$  and stability margin  $s_m$  (maximum sensitivity  $M_s = \frac{1}{s_m}$ )



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- **Stability and robustness**: gain margin  $g_m$ , phase margin  $\phi_m$  and stability margin  $s_m$  (maximum sensitivity  $M_s = \frac{1}{s_m}$ )
- Performances

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# **Stability and robustness**: gain margin $g_m$ , phase margin $\phi_m$ and stability margin $s_m$ (maximum sensitivity $M_s = \frac{1}{s_m}$ )

#### Performances

Performance specifications Expression in the frequency-domain

Time-domain: overshoot, rise time, settling time, steady state-error



9/22

#### Control System Synthesis

#### 09/09/2020 9/22

- Driving example: Cruise control
- Expression in the frequency-domain

System design considerations

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Expression in the frequency-domain

**Stability and robustness**: gain margin  $q_m$ , phase margin  $\phi_m$  and stability margin  $s_m$  (maximum sensitivity  $M_s = \frac{1}{s_m}$ )

#### Performances

- Time-domain: overshoot, rise time, settling time, steady state-error
- Frequency-domain: peak value(s), peak frequency, gain crossover frequency and bandwidth.



Expression in the frequency-domain

Driving example: Cruise control

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**Stability and robustness**: gain margin  $g_m$ , phase margin  $\phi_m$  and stability margin  $s_m$  (maximum sensitivity  $M_s = \frac{1}{s_m}$ )

#### Performances

- Time-domain: overshoot, rise time, settling time, steady state-error
- Frequency-domain: peak value(s), peak frequency, gain crossover frequency and bandwidth.

<b>P</b> (s) =	$\frac{k}{\frac{1}{\omega_0^2}s^2 + \frac{2\xi}{\omega_0}s + 1}$			
$\phi = \mathit{arccos}\xi$				

Property	Value
Steady-state value	k
Rise time	$T_r = e^{\phi/tan\phi}$
Overshoot	$M_{ m p}=e^{-\pi\xi\sqrt{1-\xi^2}}$
Settling time 2%	$T_{s} \sim rac{1}{4 \xi \omega_{0}}$

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## Performance specifications

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**Stability and robustness**: gain margin  $g_m$ , phase margin  $\phi_m$  and stability margin  $s_m$  (maximum sensitivity  $M_s = \frac{1}{s_m}$ )

#### Performances

- Time-domain: overshoot, rise time, settling time, steady state-error
- Frequency-domain: peak value(s), peak frequency, gain crossover frequency and bandwidth.

 $10^{2}$  $10^{0}$  $10^{-2}$ 

\_90

-180  $10^{-1}$ 



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(b) Frequency responses Control System Synthesis

 $10^{0}$ 

Normalized frequency  $\omega/\omega_0$ 

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 $10^{1}$ 

Expression in the frequency-domain

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Expression in the frequency-domain

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$$\mathcal{C}(s) = k_{\mathcal{P}} + rac{k_{\mathcal{P}}}{s} o \mathcal{T}(s) = rac{rac{k_{\mathcal{P}}}{k_i}s+1}{rac{m}{k_i}s^2 + rac{b+k_{\mathcal{P}}}{k_i}s+1}$$

#### Overshoot < 10%</p>

Expression in the frequency-domain

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- Overshoot < 10%
- $\xi = 0.6$



Expression in the frequency-domain

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$$\mathcal{C}(oldsymbol{s})=k_{oldsymbol{
ho}}+rac{k_{oldsymbol{
ho}}}{oldsymbol{s}}oldsymbol{
ho}+1}{rac{m}{k_{oldsymbol{
ho}}}oldsymbol{s}+2}+rac{k_{oldsymbol{
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Expression in the frequency-domain

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$$\mathcal{C}(s)=k_{
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ightarrow \mathcal{T}(s)=rac{rac{k_{
ho}}{k_{
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ho}}s^2+rac{b+k_{
ho}}{k_{
ho}}s+1}$$

• Overshoot 
$$< 10\%$$
  
 $\Rightarrow \xi = 0.6$   
• Rise time  $< 5s$   
 $\Rightarrow \omega_0 = 0.7$ 

Expression in the frequency-domain

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• 
$$\xi = 0.6$$

Rise time 
$$<$$
 5s

$$\rightarrow \omega_0 = 0.7$$

■ Steady-state error < 2%

Expression in the frequency-domain

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$$\mathcal{C}(s) = k_{
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- Overshoot < 10%</p>
  - $\xi = 0.6$
- Rise time < 5s</p>
- $\rightarrow \omega_0 = 0.7$
- Steady-state error < 2%
- ensured through the integral action

Expression in the frequency-domain

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- Overshoot < 10%</p>
  - $\xi = 0.6$
- Rise time < 5s</p>
- $\rightarrow \omega_0 = 0.7$
- Steady-state error < 2%
- ightarrow ensured through the integral action
  - Finally:  $k_p = 3600$  and  $k_i = 1450$

Expression in the frequency-domain

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$$C(s)=k_p+rac{k_p}{s}
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Overshoot < 10%</li>

 *ξ* = 0.6

Rise time 
$$< 5s$$

$$\rightarrow \omega_0 = 0.7$$

- Steady-state error < 2%
- → ensured through the integral action Finally:  $k_p = 3600$  and  $k_i = 1450$



Figure: Reference tracking

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Expression in the frequency-domain

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Overshoot < 10%</li>
 → *ξ* = 0.6

Rise time 
$$< 5s$$

$$\rightarrow \omega_0 = 0.7$$

- Steady-state error < 2%
- → ensured through the integral action Finally:  $k_p = 3600$  and  $k_i = 1450$



Figure: Disturbance rejection

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Focus on the open-loop transfer function  $L = PC \rightarrow C = L/P$  of high order



## Performance specifications Loopshaping

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- Design technique in the frequency-domain
- Focus on the open-loop transfer function  $L = PC \rightarrow C = L/P$  of high order



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#### impact of system design on feedback possibilities

## **Fundamental limitations**

System design considerations

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- impact of system design on feedback possibilities
- $\blacksquare$  unstable system  $\rightarrow$  needs a fast controller (bandwidth of sensors and actuators)

## **Fundamental limitations**

System design considerations

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- impact of system design on feedback possibilities
- unstable system  $\rightarrow$  needs a fast controller (bandwidth of sensors and actuators)
- systems with time-delays  $\rightarrow$  impossible to take fast control actions

Driving example: Cruise control

System design considerations

## **Fundamental limitations**

System design considerations

## impact of system design on feedback possibilities

- unstable system  $\rightarrow$  needs a fast controller (bandwidth of sensors and actuators)
- systems with time-delays ightarrow impossible to take fast control actions
- limitations often expressed by conditions on the poles and zeros of the system



System design considerations

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- impact of system design on feedback possibilities
- unstable system  $\rightarrow$  needs a fast controller (bandwidth of sensors and actuators)
- systems with time-delays ightarrow impossible to take fast control actions
- → limitations often expressed by conditions on the poles and zeros of the system
- Can you rework on system design while doing control?

#### Sensitivity minimization

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#### Sensitivity minimization

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# in crossover frequ



$$S = \frac{1}{1 + PC}$$

**S** is the transfer between r and the error  $\varepsilon$ 

Sensitivity minimization

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**S** is the transfer between *r* and the error  $\varepsilon$ 

Limited overshoot: 
$$M_{s} = \max_{\omega} |S(j\omega)|_{\infty} \leq \gamma$$

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Sensitivity minimization

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System design considerations

#### Sensitivity minimization

- S is the transfer between r and the error  $\varepsilon$ 
  - Limited overshoot:  $M_s = \max_{\omega} |S(j\omega)|_{\infty} \leq \gamma$
- $M_{\rm s}$  is also a measure of robustness  $\rightarrow$



 $S = \frac{1}{1 \perp PC}$ 



Sensitivity minimization

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#### Sensitivity minimization

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$$S = \frac{1}{1 + PC}$$

- S is the transfer between r and the error  $\varepsilon$
- Limited overshoot:  $M_s = \max_{\omega} |S(j\omega)|_{\infty} \leq \gamma$
- $M_{\rm s}$  is also a measure of robustness  $\rightarrow$ 
  - **Disturbance attenuation** when  $|S(\gamma \omega)|_{\infty} \leq 1$

Sensitivity minimization

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$$S = \frac{1}{1 + PC}$$

- S is the transfer between r and the error  $\varepsilon$
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- $\rightarrow M_s$  is also a measure of robustness
- $ightarrow \,\, {f Disturbance attenuation when } \left| {\cal S}(\jmath\omega) 
  ight|_\infty \leq 1$
- Low static error:  $\min_{S} \max_{\omega \in (0,\omega_c)} |S(j\omega)|$

Sensitivity minimization

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- Low static error:  $\min_{\mathcal{S}} \max_{\omega \in (0,\omega_c)} |\mathcal{S}(j\omega)|$ 
  - $\omega_c$  is the cutoff frequency

Sensitivity minimization

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$$S = \frac{1}{1 + PC}$$

- S is the transfer between r and the error  $\varepsilon$
- Limited overshoot:  $M_s = max_\omega |S(j\omega)|_\infty \leq \gamma$
- $\rightarrow M_s$  is also a measure of robustness
- $ightarrow \,\, {f Disturbance attenuation when } \left| {\cal S}(\jmath\omega) 
  ight|_\infty \leq 1$
- Low static error:  $\min_{S} \max_{\omega \in (0,\omega_c)} |S(j\omega)|$ 
  - $\omega_c$  is the cutoff frequency
  - $\square \lim_{t \to \infty} \epsilon(t) = \lim_{s \to 0} S(s)$



Bode's integral formula

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## The sensitivity function cannot be made small over a wide frequency range.

Bode's integral formula (invariant)



Bode's integral formula

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## The sensitivity function cannot be made small over a wide frequency range.

Bode's integral formula (invariant)

reducing the sensitivity at one frequency increases it at another



Bode's integral formula

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## The sensitivity function cannot be made small over a wide frequency range.

Bode's integral formula (invariant)

- reducing the sensitivity at one frequency increases it at another
- Right Half-Plane (RHP) poles in the process makes it worse



Bode's integral formula

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## The sensitivity function cannot be made small over a wide frequency range.

- Bode's integral formula (invariant)
- reducing the sensitivity at one frequency increases it at another
- Right Half-Plane (RHP) poles in the process makes it worse
- Control design is always a compromise

Bode's integral formula

Driving example: Cruise control Expression in the frequency-domain

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#### Bode's integral formula



For an internally stable closed loop system, if  $\lim_{s\to\infty} sL(s) = 0$ :

where  $p_k$  are the RHP poles of the open-loop L.



Bode's integral formula

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$$\int_{0}^{\infty} \log(|\mathcal{S}(\jmath\omega)|) d\omega = \pi \sum \mathcal{p}_k$$

where  $p_k$  are the RHP poles of the open-loop L.





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Bode's integral formula

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For an internally stable closed loop system, if  $\underset{s 
ightarrow \infty}{lim} sL(s) =$  0:

$$\int_{0}^{\infty} \log(|\mathcal{S}(\jmath\omega)|) d\omega = \pi \sum \mathcal{p}_k$$

where  $p_k$  are the RHP poles of the open-loop L. Similarly:  $\int_{-\infty}^{\infty} \frac{\log(|T(j\omega)|)}{d\omega} d\omega = \pi \sum_{k=1}^{\infty} \frac{1}{2}$ 

$$\int_0 \qquad \omega^2 \qquad \omega^2 = \pi \sum$$

where  $z_i$  are the RHP zeros of the open-loop L.

Gain crossover frequency inequality

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#### Gain crossover frequency inequality

Internal stability



$$egin{aligned} P(s) &= P_{mp}(s)P_{nmp}(s) \ & |P_{mp}(\jmath\omega)| = 1 \end{aligned}$$

$$P(s) = \frac{s-2}{(s+1)(s-1)} = \frac{s+2}{(s+1)^2} \frac{(s-2)(s+1)}{(s+2)(s-1)} = P_{mp}P_{nmp}$$

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Gain crossover frequency inequality

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$$egin{aligned} P(s) &= P_{mp}(s)P_{nmp}(s) \ & |P_{mp}(\jmath\omega)| = 1 \end{aligned}$$

$$argL(j\omega_{gc}) = argP_{nmp}(j\omega_{gc}) + argP_{mp}(j\omega_{gc}) + argC(j\omega_{gc}) \ge -\pi + \phi_m$$

 $\phi_m$  is the desired phase margin

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## Fundamental limitations Gain crossover frequency inequality

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$$egin{aligned} P(s) &= P_{mp}(s)P_{nmp}(s) \ & |P_{mp}(arphi\omega)| = 1 \end{aligned}$$

 $argL(j\omega_{gc}) = argP_{nmp}(j\omega_{gc}) + argP_{mp}(j\omega_{gc}) + argC(j\omega_{gc}) \ge -\pi + \phi_m$  $\phi_m$  is the desired phase margin

Assuming that *C* has no poles or zeros in the RHP:

$$arg P_{mp}(\jmath \omega_{gc}) + arg \mathcal{C}(\jmath \omega_{gc}) = slope * rac{\pi}{2}$$

## Fundamental limitations

Gain crossover frequency inequality

## Performance specifications

Driving example: Cruise control Design trade-offs Expression in the frequency-domain Loopshaping

#### Fundamental limitations

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$$\textit{argP}_{\textit{nmp}}(\jmath\omega_{\textit{gc}}) + \textit{slope} * rac{\pi}{2} \geq -\pi + \phi_{\textit{m}}$$

- $\rightarrow$  there is a trade-off between phase margin and speed
- Fast RHP poles  $\rightarrow$  larger  $\omega_{gc}$
- Slow RHP zeros  $\rightarrow$  lower  $\omega_{gc}$

#### Internal stability

#### Performance specifications

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#### Fundamental limitations

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#### Internal stability = stability of the gang of four

## **Fundamental limitations**

Internal stability

## Performance specifications

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#### Fundamental limitations

- System design considerations Sensitivity minimization Bode's integral formula Gain crossover frequency inequali Internal stability
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## **Fundamental limitations**

#### Internal stability

### Performance specifications

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- System design considerations Sensitivity minimization Bode's integral formula Gain crossover frequency inequalit Internal stability
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$$\left\{\begin{array}{c}T(p_k)=0\\S(p_k)=1\end{array}\right\}\left\{\begin{array}{c}T(z_i)=1\\S(z_i)=0\end{array}\right.$$

Internal stability

## Performance specifications

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System design considerations Sensitivity minimization Bode's integral formula Gain crossover frequency inequa Internal stability

- Internal stability = stability of the gang of four
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$$\begin{cases} T(p_k)=0 \\ S(p_k)=1 \end{cases} \begin{cases} T(z_i)=1 \\ S(z_i)=0 \end{cases}$$

#### **Maximum modulus principle:** if *G* is bounded and analytic in the RHP:

$$\max_{\omega \in \mathbb{R}} |G(\jmath \omega)| = \max_{\mathsf{Re}(s) \geq 0} |G(s)|$$

#### Maximum modulus principle

#### Performance specifications

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#### Fundamental limitations

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(a) Requirements for sensitivity

(b) Requirements for complementary sensitivity

$$S_r(s) = rac{M_s s}{s+a}$$
  $T_r(s) = rac{M_t b}{s+b}$ 

Pauline Kergus - Karl Johan Åström

#### Maximum modulus principle

## Performance specifications

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(a) Requirements for sensitivity

(b) Requirements for complementary sensitivity

$$S_r(s) = rac{M_s s}{s+a} \qquad T_r(s) = rac{M_t b}{s+b}$$

For the sensitivity:

$$1 \geq rac{|S(\jmath\omega)|}{|S_r(\jmath\omega)|} \geq rac{|S(z)|}{|S_r(z)|} = rac{z+a}{M_s z}$$

Pauline Kergus - Karl Johan Åström

#### Maximum modulus principle

### Performance specifications

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$$a \leq z(M_s - 1) 
ightarrow \omega_{sc} \leq z \sqrt{rac{M_s - 1}{M_s + 1}}$$

Pauline Kergus - Karl Johan Åström

## Conclusions

## Performance specifications

Driving example: Cruise control Design trade-offs Expression in the frequency-domain Loopshaping

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 $\blacksquare$  The different specifications are linked through the gang of four  $\rightarrow$  trade-offs

## Conclusions

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different criterias in time-domain and frequency-domain



## Conclusions

#### Performance specifications

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#### Fundamental limitations

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- The different specifications are linked through the gang of four ightarrow trade-offs
- different criterias in time-domain and frequency-domain
- limitations due to the RHP poles and zeros and time-delays of the process