Sporadic Event-Based Control of First-Order Linear Stochastic Systems

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Abstract

The standard approach in computer-controlled systems is to sample and control periodically. In certain applications, such as networked control systems or energy-constrained systems, it could be advantageous to instead use event-based control schemes. Aperiodic event-based control of first-order stochastic systems has been investigated in previous work. In any real implementation, however, it is necessary to have a well-defined minimum inter-event time. In this paper, we explore two such sporadic control schemes for first-order linear stochastic systems and compare the achievable performance to both periodic and aperiodic control. The results show that sporadic control can give better performance than periodic control in terms of both reduced process state variance and reduced control action frequency.

Key words: Event-based control; Stochastic optimal control; Real-time systems; Networked control systems; Hybrid systems

1 Introduction

Digital feedback controllers are most often implemented using periodic sampling, computation, and actuation. This approach enables the control designer to utilize standard sampled-data system theory or to discretize a continuous-time controller assuming a fixed sampling rate and constant hold intervals [3].

For some applications, however, event-based control schemes may have an advantage over periodic schemes. In networked control applications, it could make sense to only transmit information when something significant has occurred in the system, in order to save bandwidth. In embedded applications, it may be essential to minimize the number of control actions in order to save energy. In the application of inventory control it seems rational to replenish stock only when it is low rather than on a periodic basis, if there is a fixed transportation cost. Some sensors such as rotary motion encoders only give new measurements at ahead-of-time unknown events.

Event-based control as a technology is of course not new. Mostly, however, it has been applied in an ad-hoc way. This can be attributed to the lack of a comprehensive theory, which in turn can be explained by the mathematical difficulties involved. A discrete-time formulation can sometimes make it slightly easier to obtain a solution. Some recent papers have thus solved optimal discrete-time estimation problems, with limited [8] or event-triggered [5] measurements.

From a control-theoretic point of view, event-based control systems can be viewed as hybrid systems. In this paper, we consider first-order linear stochastic systems, where an exogenous random disturbance (modelled as white noise) causes the process state to drift. The control law generates discrete events when the state crosses certain boundaries. Hence, our system falls into the category of stochastic hybrid systems as defined in [7].

Event-based control of first-order linear stochastic systems was studied in [2]. It was shown that, compared to periodic control, the output variance could be significantly reduced assuming the same mean time between events. The control was realized by applying an impulse action whenever the magnitude of the system state exceeded a certain threshold. This work was elaborated in [9], which explores, among other things, event-based control with piecewise constant control signals and level-triggered sampling.

From a real-time systems point of view, however, tasks triggered by asynchronously generated events cannot be guaranteed service unless there is a well-defined minimum inter-arrival time. For the controller presented in [2] there was no such minimum inter-arrival time. In accordance with real-time systems terminology [4], we will refer to such a control policy as aperiodic.

In this paper, we explore the class of sporadic event-
based controllers for first-order linear stochastic systems. With a minimum inter-arrival time $T$ between events, such a controller can be guaranteed not to consume more than a certain network bandwidth or CPU utilization. Two sporadic controllers will be studied. The first controller measures the process state continuously and can take control actions at any time, but no more often than every $T$ seconds. The second controller measures the process state every $T_s$ seconds until a control action is applied, and resumes measurements $T$ seconds after the last control action.

2 Problem Formulation

The process to be controlled is given by the linear stochastic differential equation

$$dx = ax dt + ud t + \sigma dw, \quad x(0) = 0,$$

where $x$ is the state, $u$ the control signal, $w$ is a Wiener process with unit increment variance, $a$ is the pole of the process, and $\sigma > 0$ is the intensity of the process noise. The control signal is zero except at events $t_k$, when it is allowed to be a Dirac pulse of magnitude $u_k$:

$$u(t) = \sum_{k=0}^{\infty} \delta(t - t_k)u_k.$$

The controller chooses when to generate an event based on the state of the system. After each event there is a period of inactive state of duration $T$, when no new events can be generated, see Fig. 1.

The performance is measured by the stationary state cost,

$$J_x = \lim_{t \to \infty} \sup_{t'} \frac{1}{T} \int_0^T x^2 ds,$$

and by the average control rate (or control cost),

$$J_u = \lim_{t \to \infty} \sup_{t'} \frac{1}{T} N_u(0, t),$$

where $N_u(t_1, t_2)$ is the number of control actions in the interval $(t_1, t_2)$. The total cost to be minimized is

$$J = J_x + \rho J_u,$$

where $\rho \geq 0$ is the relative cost of control actions.

Normalized Formulation

To reduce the number of free parameters we can use coordinate scaling to fix $\sigma = T = 1$. The parameters that remain, $a$ and $\rho$, suffice to specify the problem up to coordinate scaling, and the original variables can be retrieved from inverse scaling. The parameters $\sigma$ and $T$ will be kept in the presentation when they add insight.

Let the transformed variables be described by

$$dt = Td\tau, \quad dw = \sqrt{T} dv, \quad x = \sigma \sqrt{T} x'.$$

The dynamics become

$$dx' = a' x' d\tau + u' d\tau + dw,$$

where $u' = \sqrt{T} \sigma^{-1} u$, and $a' = aT$ is the relevant measure of process speed. The original costs are retrieved as

$$J_x = \sigma^2 T J'_x, \quad J_u = T^{-1} J'_u,$$

so $\rho' = \frac{\rho}{\sigma^2 T^2}$ is the proper weighting after normalization. The normalized problem is described by the parameters

$$a' = aT, \quad \rho' = \frac{\rho}{\sigma^2 T^2}.$$

3 Sporadic Control

3.1 General Observations

A sporadic controller is defined by two properties: when it generates an event and what control signal is used at the event. It is easy to see that an optimal controller for the problem above must satisfy the following:

- At any event $t_k$, the control signal $u_k$ is chosen to bring $x$ to the origin, i.e. $u_k = -x(t_k)$. 
- When an event is permitted, the decision of whether to generate one is a function only of the state $x$, and due to symmetry only of the absolute value $|x|$. 
- If an event should be generated when $|x| = r$, one should also be generated whenever $|x| \geq r$.

Thus, the optimal control policy is a threshold policy where an event is triggered to bring $x$ to zero whenever permitted and $|x| \geq r$.

To find the optimal threshold $r$, we evaluate $J$ as a function of $r$ in the closed loop system. To facilitate this, we first consider what happens between events.

3.2 Evolution Between Events

Between events the control signal is known, and the system evolves as a linear stochastic process. Assume that an event occurs at time $t_k = 0$, and that we want to predict the evolution from that time, from the state prior to the event $x_0 = x(0-)$. Let

$$m(t) = E(x(t)), \quad P(t) = E(x(t)^2) - m(t)^2$$

be the expected state trajectory and the expected state variance due to process noise entering after the event respectively, with initial conditions

$$m(0) = x_0 + u_k, \quad P(0) = 0.$$

The distribution of $x(t)$ will be Gaussian with mean $m(t)$ and variance $P(t)$.

The expected state cost during the interval $(0, t)$ can be expressed as the sum of one contribution $V_P(t)$ from $P$ and one $V_m(t)$ from $m$ according to

$$\int_0^t E(x(s)^2) ds = \int_0^t (P(\tau) + m(\tau)^2) d\tau = V_P(t) + V_m(t).$$

Since there is no feedback between events, $u$ will enter the evolution only through $m(t)$. We find that

$$E(dP) = E(2xdx + dw^2) - 2mE(dn)$$

$$= E(2x(ax dt + ud t + \sigma^2 dw^2) - 2m(am + u)$$

$$= (2aP + \sigma^2)dt.$$
Starting from \( P(0) = 0 \), the solution is

\[
P(t) = \begin{cases} 
1 - e^{a t} & a \neq 0, \\
\frac{a t}{2}, & a = 0.
\end{cases}
\]  

(4)

Integrating, the process noise contribution to the state cost during the interval \((0, t)\) is

\[
V_p(t) = \begin{cases} 
\frac{a^2}{2} \left( t - \frac{e^a t - 1}{2} \right), & a \neq 0, \\
\frac{a^2}{4} t^2, & a = 0.
\end{cases}
\]  

(5)

The expected trajectory evolves according to \( E(dm) = E(dx) = (am + u) dt \), giving the prediction

\[
m(t) = e^{a t} m(0) + \int_{0}^{t} e^{a(t-\tau)} u(\tau) d\tau.
\]

With no control during the interval \((0, t)\), the cost is

\[
V_m(t) = \int_{0}^{t} m(s)^2 ds = Q(t) m(0)^2,
\]

\[
Q(t) = \left( \frac{e^{a t} - 1}{2a}, \quad a \neq 0, \right)
\]

\[
\left( t, \quad a = 0. \right)
\]

(6)

### 3.3 Sporadic Control with Continuous Measurements

We assume that the process state is measured continuously in the active state. As soon as the state leaves the region \( |x| < r \), an event is generated and the controller is put in the inactive state for an interval of length \( T \).

Since the system is reset to the same state at each event, the expected cost and time from one event to the next are enough to find the stationary costs, as

\[
J_x = \frac{V_{\text{active}} + V_{\text{inactive}}}{T_{\text{active}} + T}, \quad J_a = \frac{1}{T_{\text{active}} + T},
\]

where \( V_{\text{active}} \) and \( T_{\text{active}} \) are the expected state costs and dwell times during one period of active state, and \( V_{\text{inactive}} = V_p(T) \). We will characterize the behavior between two events by modifying the system so that it starts at one event and is stopped at the next.

The expected cost and dwell time during one period of active state can be found as

\[
V_{\text{active}} = \int x^2 F(x) dx, \quad T_{\text{active}} = \int F(x) dx,
\]

(7)

where \( F(x) = \int_{-\infty}^{x} f(x, t) dt \) is the accumulated state density of the density \( f(x, t) \) in the active state.

The system enters the active state as

\[
f(x, t = 0) = \begin{cases} 
\varphi(x), & |x| < r \\
0, & |x| \geq r
\end{cases}
\]

where \( \varphi(x) \) is Gaussian with zero mean and variance \( P(T) \). The time evolution is given by the Fokker-Planck equation (see e.g. [1], [6]) (with \( \sigma = 1 \)):

\[
\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (f a^2) - \frac{\partial}{\partial x} (f a x) = \frac{1}{2} \frac{\partial^2 f}{\partial x^2} - ax \frac{\partial f}{\partial x} - af,
\]

with absorbing boundary conditions \( f(\pm r, t) = 0 \). Since \( f(x, t) \to 0 \) as \( t \to \infty \) we can integrate over \( t \in [0, \infty) \) to find a differential equation for \( F(x) \):

\[
-\varphi(x) = \int_{0}^{\infty} \frac{\partial f}{\partial t} dt = \frac{1}{2} F''(x) - ax F'(x) - a F(x),
\]

with boundary conditions \( F(\pm r) = 0 \). The solution exists as long as \( \varphi(x) \) does, and can be found numerically with a linear ODE Boundary Value Problem (BVP) solver or analytically as

\[
F(x) = 2 \int_{y = -r}^{x} e^{a(x^2 - y^2)} \int_{0}^{y} \varphi(z) dz dy, \quad |x| \leq r.
\]

(8)

Fig. 2 shows the costs as a function of \( r \) for the case \( T = \sigma = 1 \) and \( a \in \{-0.5, 0, 0.5\} \). Other cases can be reconstructed by scaling as explained in Section 2. We see an initial decrease in the state cost as the threshold is increased, so the optimal threshold is non-zero even when \( r = 0 \). We also see that both costs decrease as \( a \) decreases, since the system becomes easier to control.

The cost functions can alternatively be found from

\[
V_{\text{active}} = \int \varphi(x) V_x(x) dx, \quad T_{\text{active}} = \int \varphi(x) \theta(x) dx,
\]

(9)

where \( V_x(x) \) is the expected state cost until the next event starting in the active state at \( x \), and \( \theta(x) \) is the corresponding expected dwell time (or first passage time, see [6]). The value function \( V(x) = V_x(x) - J \theta(x) \) can be used for dynamic programming.

When \( x = \pm r \), \( V_x(x) = \theta(x) = 0 \), and when \( |x| < r \),

\[
E(d V_x(x)) = -x^2 dt, \quad E(d \theta(x)) = -1 dt
\]

which together with the dynamics (1) gives

\[
-x^2 dt = E(V_x' dx + \frac{1}{2} V_x'' dx^2) = (ax V_x' + \frac{1}{2} V_x') dt,
\]

for \( V_x(x) \), and similarly for \( \theta(x) \). The solutions can be found numerically with an ODE BVP solver, or as

\[
\left( \frac{V_x(x)}{\theta(x)} \right) = 2 \int_{y = x}^{y = r} e^{a(x^2 - y^2)} \left( \frac{z^2}{1} \right) dz dy.
\]

(10)

We note that problem can be extended in a few ways that fit well with our solution methods. Behavior in the inactive state only affects the solution through \( T_{\text{inactive}}, V_{\text{inactive}}, \) and the state density when entering the active state \( \varphi(x) \). Possible extensions include a delay \( \tau \leq T \) from the issue of an event to the actuation of the control impulse, and a stochastically varying inactive time \( T \).

### 3.4 Sporadic Control with Discrete Measurements

We now assume that the process is sampled with the interval \( T_s \leq T \) in the active state. Any deviations of the state outside the threshold between samples will go unnoticed. As before, when a deviation is detected at time \( t_k \), the controller issues a control event and enters the inactive state, where it stays for \( T \) seconds. We now let \( \{t_k\} \) denote all sampling instants, which progress as

\[
t_{k+1} = \begin{cases} 
t_k + T_s, & |x_k| < r, \\
t_k + T, & |x_k| \geq r.
\end{cases}
\]

To find the optimal threshold \( r \), the cost is characterized as a function of \( r \). To this end, we compute the
stationary state distribution (see [6]) at the sampling instants. Between sampling instants, the state evolves as

\[
x_{k+1} = \begin{cases} \exp(aT)x_k + w_k(T), & |x_k| < r, \\ w_k(T), & |x_k| \geq r, \end{cases}
\]

(11)

where \(w_k(t)\) is a Gaussian random variable with zero mean and variance \(P(t)\). The stationary density always exists since there is a positive probability to go from any state \(x\) to any state interval \((x_1, x_2)\) in one step, and for \(|x| \geq r\) the density after any time step falls off as a Gaussian with variance \(P(T)\). The accumulated state cost from time \(t_k\) to time \(t_{k+1}\) is given by

\[
V_{\text{stay}} = Q(T_s) \mathbb{E}\left\{x_k^2 \mid |x_k| < r\right\} + V_P(T_s)
\]

(12)

if the controller stays in the active state and by

\[
V_{\text{exit}} = V_P(T)
\]

(13)

if the controller enters the inactive state.

Finally, assuming stationarity, the costs become

\[
J_x = \frac{p_{\text{stay}} V_{\text{stay}} + p_{\text{exit}} V_{\text{exit}}}{p_{\text{stay}} T_s + p_{\text{exit}} T} \quad J_u = \frac{p_{\text{exit}}}{p_{\text{stay}} T_s + p_{\text{exit}} T}
\]

where

\[
p_{\text{stay}} = \text{Prob}\{|x_k| < r\} = 1 - p_{\text{exit}}.
\]

The stationary distribution of \(x_k\) can be found numerically by discretizing the state space and then iterating the distribution according to (11) until convergence.

Fig. 3 shows the costs as a function of \(r\) for the case \(T = T_s = \sigma = 1\) and \(a \in \{-0.5, 0, 0.5\}\). Here, the state cost increases monotonically with \(r\). With \(T_s < T\) we would have an initial decrease, approaching the behavior for continuous measurements as \(T_s/T \to 0\). The control action frequency \(J_u\) falls off faster with increasing threshold than for the continuous measurement case, since \(x\) is checked against the threshold less often with discrete measurements. As expected, both costs decrease with \(a\).

Alternatively, the expected state cost \(V_u(x)\) and dwell time \(\theta(x)\) until the next event starting from state \(x\) can be iterated until convergence. As in the continuous case, we could extend the problem formulation with actuation delay and stochastically varying inactive time.

4 Comparison of Control Schemes

Sporadic control with continuous and discrete measurements (with \(T_s = T\)) will now be compared to periodic and aperiodic control. We first discuss how to make the comparison.

4.1 Periodic and Aperiodic Control

An aperiodic controller sets the process state \(x\) to zero whenever \(|x| \geq r\) using an impulse control action \([2]\). The cost functions can be found by letting \(\varphi(x)\) approach a unity Dirac pulse in (8) or (9), yielding

\[
J_x = V_{\text{active}}/T_{\text{active}}, \quad J_u = T_{\text{active}}^{-1}.
\]

We assume that periodic control is also implemented with impulse control action, such that \(x\) is periodically reset to zero. The sampling interval is restricted to be no shorter than for the sporadic schemes. The costs become

\[
J_x = V_P(T)/T, \quad J_u = T^{-1}.
\]

(14)
4.2 Preliminaries

For the sporadic controllers, minimization of the loss function $J$ for a given $\rho$ determines an optimal threshold $r$, which maps to an optimal average event rate $J_a$. The same holds for aperiodic control. In periodic control, however, there is no threshold. Instead, $\rho$ determines the optimal sampling interval. Hence, we can parametrize controllers from all four classes by average event rate.

The four controllers differ by the constraints on when they can generate control events. A scheme with fewer restrictions will be harder to implement but give a lower cost $J_a$. As $\rho$ decreases when $r = 0$, aperiodic control should approach aperiodic since $T$ becomes negligible. When $\rho = 0$, sporadic control with discrete measurements and $T_a = T$ approaches the periodic since there remains no incentive to omit an event.

When $a < 0$, $J_a$ and therefore $J$ is bounded by the variance achieved in open loop. As $\rho$ increases, all controllers will generate fewer events so that $J_a \to 0$, and ultimately $J_a$ will approach a maximum. The limit can be found from (14), where $J_a \to -1/2a$ as $T \to \infty$.

4.3 Comparison

The trade-off between state variance and average event frequency is made explicit in Fig. 4, where $J_a$ is plotted against $J_a$ for the four controllers. The results for $\sigma \neq 1$ are found by scaling $J_a$ by $\sigma^2$. It is seen that the controllers are strictly ranked in performance by how much freedom they have to generate events, and that the sporadic controller with discrete measurements always outperforms the periodic one.

Fig. 4 also shows what we consider the main advantage of event-based control: fewer events are needed for the same state cost. With periodic control, the variance increases quite rapidly with lower sampling rate. However, with sporadic control the average control rate can be reduced much further without the same penalty. For example, when $a = 0.5$ the average cost rate may be decreased by about 40% for only slightly more variance, using sporadic control with discrete measurements.

A notable result is that for sporadic control with continuous measurements, $J_a$ can be made somewhat smaller with fewer events. This is also seen in the upper plot of Fig. 2, where $J_a$ attains a minimum for $r > 0$. Apparently, there is a hidden cost in issuing a control event, due to the risk that large state errors will arise while in the inactive state. This phenomenon is absent for discrete measurements and $T_a = T$ since in this case events are generated independent of past actions.

Fig. 5 shows the optimal achievable cost $J^*$ for the four controllers. It is notable that for the stable system $a = -0.5$ the optimal periodic controller chooses to never sample when $\rho \geq 1$, while the sporadic controllers just raise their thresholds and remain ready to deal with large disturbances.

5 Higher Order Systems

So far, we have only considered first order systems. When raising the state dimension, there are many different generalizations worthy of study, depending on which is the constraining resource that motivates using event based control. We will briefly discuss some possibilities.

5.1 Formulations

The dynamics and cost $J_a$ are naturally extended to
\[ dx = Ax dt + Bu dt + \zeta, \]
\[ J_a = \lim_{t \to \infty} \sup_{\zeta} \frac{1}{t} \int_0^t \eta^T Q \eta ds, \]
where $x, w$ and possibly $u$ are vectors. One natural generalization of the measurement equation is
\[ dy = C x dt + d w_m, \]
where $d w_m$ is measurement noise.

The possible forms of the controller, actuators, and sensors are more varied. Some scenarios are:

- **Communication constraints.** Events are packets sent over a communication channel, from an observer at the sensor to a controller at the actuator. The observer decides when to send a state estimate to the controller, which predicts the plant state in open loop in between. Each event resets the prediction error.

- **Actuator constraints.** The actuator only generates pulses of certain shapes, with some cost per pulse. The controller plans for an optimal and possibly long sequence of pulses, which is sensitive to timing.

- **Sensor constraints.** The sensor only gives measurements under some conditions, e.g. at or beyond some thresholds. The control problem becomes a state estimation problem with nonstandard measurement information, for which the Kalman Filter is not optimal.

- **Processing power constraints.** A simple control law is needed. The best bet is probably to postulate one and optimize over a few parameters.

We can consider a single control loop, or multiple loops sharing the same limited resource. The loops can be independent, or cooperate to control a single plant. It seems unreasonable that the controllers should know each others’ state, especially with communication constraints.

5.2 Methods

The discretizations applied in this paper can be generalized to higher state dimensions, but become impractical beyond a few states due to the curse of dimensionality. Sometimes the dimension can be reduced somewhat; e.g. if the state is estimated with a stationary Kalman Filter, the distribution of the actual state is known conditioned on the estimate. Otherwise, nonlinear process dynamics come at a modest additional cost. Optimal stochastic control is in principle applicable to both the communication and actuation constrained scenarios.

Beyond a few states, simpler formulations are necessary for a solvable problem. This may include reducing the amount of uncertainty in the problem. A communication constrained problem easily becomes pointless with too little uncertainty, while an actuator constrained problem may still preserve its major features.

6 Conclusions

In some applications there is a cost related to the execution of a control signal, regardless of the magnitude of that signal. If that cost is included in the performance
objective of the controller, it will be meaningful to reduce the frequency of control actions. This may be accomplished with a periodic controller by lengthening the sampling interval. However, the penalty in terms of increased process state variance is significant. Trying to improve the tradeoff by not acting on small state errors naturally leads to the notion of event-based control.

In this paper, we have shown that sporadic control can provide a better tradeoff between control objectives as well as better overall control performance than periodic control, when there is a fixed cost of control actions. It is noted that the average frequency of control events can be reduced with only a small increase in state variance. Moreover, we show that sporadic control can actually reduce both the average frequency of control events and the state variance simultaneously. When the objective is to reduce the frequency of events as well as the state variance, the sporadic control schemes presented here even perform almost as well as aperiodic control, while respecting a prespecified shortest inter-event time.

Event-based control has an additional threshold parameter that should scale with the size of disturbances. If they are bigger than expected, the control approaches periodic control. If they are smaller, the threshold will act as a tolerable margin of error. Both responses are reasonable in the face of a mismatched disturbance intensity.

Obviously, to implement sporadic control where periodic control is currently used requires some changes. Unless the hardware supports continuous measurements, discrete measurements are an easier option and approach the continuous performance quite fast if one can measure more often than control. The change from periodic to sporadic control with the same measurement and control interval should require minimal modifications.

References