The Jitter Margin and Its Application in the Design of Real-Time Control Systems

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Joint work with B. Lincoln, J. Eker, K.-E. Årzén, and G. Buttazzo
Outline

1. Inverted pendulum example
2. The jitter margin
3. Best-case response times under EDF
4. Example revisited
Inverted pendulum example

Suppose you want to control three inverted pendulums using one CPU:

![Diagram of three inverted pendulums controlled by a single CPU with RTOS](image-url)
Design

- Discrete-time LQG controllers
- Sampling intervals: \((T_1, T_2, T_3) = (10, 14.5, 17.5)\) ms
- Assumed execution time: \(C_i = 3.5\) ms
- Controllers designed assuming delay of 3.5 ms
- Schedulable under both RM and EDF (with \(D_i = T_i\))
TrueTime simulations

[Henriksson and Cervin, 2002]
Delay margins under RM

Classical delay margin $L_m$: The longest delay a control loop can tolerate without becoming unstable.

(Continuous-time: $L_m = \varphi_m/\omega_c$)

<table>
<thead>
<tr>
<th>Task</th>
<th>$T$</th>
<th>$C$</th>
<th>$R$</th>
<th>$L_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>3.5</td>
<td>3.5</td>
<td>9.8</td>
</tr>
<tr>
<td>2</td>
<td>14.5</td>
<td>3.5</td>
<td>7</td>
<td>12.5</td>
</tr>
<tr>
<td>3</td>
<td>17.5</td>
<td>3.5</td>
<td>14</td>
<td>14.6</td>
</tr>
</tbody>
</table>

$\forall i : R_i < L_{mi}$

So why not stable??
The jitter margin

The delay margin is only valid for *constant* delays.

For real-time control systems, we need a *jitter margin*.

Assumptions:

- Periodic sampling (hardware/interrupt-driven)
- Time-varying input-output delay
  - \( L \) – constant delay
  - \( J \) – output jitter
A stability theorem

[Kao and Lincoln, 2004]

Continuous-time plant $P(s)$
Discrete-time controller $K(z)$
Closed-loop system assumed stable for zero delay
Freely time-varying delay $\Delta \in [0, J]$
A stability theorem

The closed-loop system is stable if

\[
\left| \frac{P_{\text{alias}}(\omega)K(e^{i\omega})}{1 + P_{\text{ZOH}}(e^{i\omega})K(e^{i\omega})} \right| < \frac{1}{\bar{J}|e^{i\omega} - 1|}, \quad \forall \omega \in [0, \pi],
\]

Continuous–time plant, discrete–time controller

![Graph showing the stability bound and the control system response](image)
Both the constant delay $L$ and the jitter $J$ contribute to the destabilization of the system.

We define the *jitter margin* $J_m(L)$ as the largest jitter for which closed-loop stability is guaranteed for any time-varying delay $\Delta \in [L, L + J_m(L)]$.

**Straightforward application of the stability theorem:**

- Include $L$ in the plant description
- Solve for the maximum allowable $J$
Jitter margin – example

\[ J_m(L) \text{ for pendulum controller 3:} \]

\[ L_m = 14.6 \]
\[ J_m(3.5) = 8.1 \]
Properties

- \( J_m(L_m) = 0 \)
- \( J_m(L) \leq L_m, \quad \forall L \)
- \( J_m(L) + L \) is an increasing function of \( L \)
Deadline assignment

Stability of the closed-loop system can be guaranteed by assigning the relative deadline

\[ D = J_m(L) + L \]

(Shorter deadlines can be assigned to guarantee an additional “phase margin” – see the proceedings)
Computing the delay and jitter

The constant delay and the jitter can be computed as

\[ L_i = R_i^b \]
\[ J_i = R_i - R_i^b \]

- \( R_i^b \) – best-case response time of task \( i \)
- \( R_i \) – worst-case response time of task \( i \)
Remarks

- The analysis should ideally include task offsets.
- If offsets are unknown, assuming worst-case phasing for $R$ and best-case phasing for $R^b$ is pessimistic.
## Response-time analysis

Available exact analysis:

<table>
<thead>
<tr>
<th></th>
<th>$FP$</th>
<th>$EDF$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Best-Case</strong></td>
<td>Redell &amp; Sanfridson (2002)</td>
<td>–</td>
</tr>
</tbody>
</table>
A lower bound under EDF

To guarantee stability, it is sufficient to have a lower bound on the best-case response-time.

- **Trivial lower bound:**
  \[ R_b^i = C_i^b \]

- **Interference analysis gives the following lower bound:**
  \[ R_i^b = C_i^b + \sum_{\forall j: D_j < R_i^b} \left[ \min \left\{ R_i^b, \frac{D_i - D_j}{T_j} \right\} - 1 \right] C_j^b \]

(Does not take “initial interference” into account)
Evaluation of the bound

- CPU loads between 0.5 and 0.99
- For each load: 100 random task sets with 2–10 tasks with random periods and execution times
- Compute mean($R_i^b / C_i$) for the longest-period task
- Compare:
  - Trivial lower bound $R_i^b = C_i^b$
  - Minimum response time found in simulation
  - Our lower bound
Evaluation of the bound

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The pendulum example – RM

- Calculate $R$ and $R^b$ for each task
- Compute the jitter margin $J_m(R^b)$ for each task
- $J < J_m(R^b) \Rightarrow$ Stable

<table>
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<tr>
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<th>$R$</th>
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<th>$J$</th>
<th>$J_m(R^b)$</th>
<th>Stable</th>
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<tbody>
<tr>
<td>1</td>
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<td>3.5</td>
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<td>4.4</td>
<td>Yes</td>
</tr>
<tr>
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<td>7.0</td>
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<td>6.4</td>
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<tr>
<td>3</td>
<td>14.0</td>
<td>3.5</td>
<td>10.5</td>
<td>8.1</td>
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The pendulum example – EDF

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(In general, EDF distributes the jitter more evenly than RM)
Conclusions

- Dynamic scheduling algorithms introduce jitter
- The *jitter margin* can be used to guarantee the stability of control tasks with output jitter
- Best-case response-time analysis useful
  - Simple lower bound under EDF presented

Future work:

- Sampling jitter
- Improved jitter analysis
- Control–scheduling co-design
Acknowledgments

- EU/ARTIST
- SSF/FLEXCON