Estimation and Direct Adaptive Control of Delay-differential Systems

ROLF JOHANSSON†

Model matching of 2-D systems and parameter identification are combined for adaptive control of delay-differential systems.

Key Words—Adaptive control; delays; discrete systems; estimation.

Abstract—The problem of adaptive prediction and direct adaptive control for systems described by delay-differential systems is considered. A model reference approach to direct adaptive control is shown to be feasible for a class of 2-D systems. A reparametrization of the 2-D system is given and the identification problem is converted into a linear estimation problem. Some necessary conditions for existence of solutions to the adaptive prediction and direct adaptive control problem are also given.

INTRODUCTION

The problem of direct adaptive control and adaptive filtering for systems described by delay-differential equations is considered. This kind of system description is naturally obtained for processes with flows, transport delays, and refluxes, (Morse, 1976; Emre et al., 1982). A quite different application described by similar system equations is image processing systems. This latter application will however not be considered here.

A model reference approach to direct adaptive control is shown to be feasible for a class of 2-D systems. A reparametrization of the 2-D system description is given and the identification problem is converted into a linear estimation problem. Some necessary conditions for existence of solutions to the adaptive prediction and direct adaptive control problem are also given.

PRELIMINARIES

Let $R[z]$ denote the ring of polynomials in $z$ with real coefficients. Here systems defined over $R[z_1,z_2]$, i.e. the ring of polynomials with real coefficients in two indeterminates — $z_1$ and $z_2$, will also be considered. Such systems are for example continuous-time systems with delays and 2-D image processing filters. The systems may as well be described by polynomials in $R(z_1)[z_2]$ or $R(z_1)[z_2]$, i.e. polynomials in $z_2$ whose coefficients are rational functions and polynomials in $z_1$, respectively. For the theory of polynomials of several variables, see Kamen and Rouche (1983) or Van der Waerden (1970). The algebraic properties of $R[z_1,z_2]$ are different from those of $R[z]$. The ring of polynomials $R[z]$ is e.g. a Euclidean domain, while $R[z_1,z_2]$ is not, see Shapiro (1975).

Traditional methods to solve model matching problems in linear systems theory, see Åström and Wittenmark (1984) or Kailath (1980) are based on the Diophantine equation

$$AR + BS = P,$$

which for given $A, B, P \in R[z]$ has solutions $R, S \in R[z]$ if $A$ and $B$ are coprime. In the 2-D case there are solutions $R, S \in R(z_1)[z_2]$ for $A, B, P \in R(z_1)[z_2]$ when $A$ and $B$ are coprime, i.e. when the Bezout identity exists, see Morf et al., (1977). There are also solutions $R, S \in R(z_1)[z_2]$ for coprime $A, B \in R(z_1)[z_2]$ and $P \in R(z_1)[z_2]$, see Khargonekar and Sonntag (1982) and theorem 2.11 of Emre et al. (1982). For further investigations on the coprimeness problem of 2-D polynomials, see Morf et al. (1977).

PROBLEM FORMULATION

The problem of direct adaptive control of 2-D systems will now be considered. The treatment is based on a model reference approach to direct adaptive control. A closer account is given, e.g. in Åström and Wittenmark (1984) or Landau (1979). The model reference approach is closely connected with the problem of exact model matching. Treatments of the 2-D case of model matching are found in e.g. Eising and Emre (1979) and Sebek (1983).
The problem is to find a continuous-time regulator described by
\[ u = \frac{S}{R} y + \frac{T}{R} u_t \] (2)
to control a continuous-time object described by the factorized input-output model
\[ y = \frac{B}{A} u, \] (3)
with input \( u \), output \( y \) and command signal \( u_c \). The problem of existence of polynomial factorizations is treated in Khargonekar (1982). Here \( A, B, R, S, T \) are polynomials with unknown real coefficients in some operator(s). The operators may be, e.g., the differential operator \( p \) and some time delay operator. In this paper the following operators will be used
\[ z_1 = \frac{a}{p + a} \quad \text{and} \quad z_2 = e^{-\tau p} \] (4)
for some given, positive constant \( a \). The operator \( z_1 \) is a low pass filter operator and \( z_2 \) represents a time delay \( \tau \). These operators are chosen because they are causal, stable operators with finite gain. Pernebo (1980) has developed an algebraic systems theory partly based on such operators and Johansson (1983) has demonstrated their relevance for adaptive control. Notice also that the transformation results in new parameters which are linear with respect to the original parameters.

The polynomials \( A \) and \( B \) are then
\[
A(z_1, z_2) = \sum_{i=0}^{n_A} \sum_{j=0}^{n_A} a_{ij} z_1^i z_2^j; \quad a_{00} = 1
\]
\[
B(z_1, z_2) = \sum_{i=0}^{n_B} \sum_{j=0}^{n_B} b_{ij} z_1^i z_2^j
\] (5)
for some finite orders \( n_A, n_{ij}, n_B \). The condition on \( a_{00} \) is imposed for the purpose of normation and is possible if the transfer function of (3) is both causal and proper. The transfer operator of (3) may also be factorized as
\[
\begin{cases}
A(z_1, z_2) \zeta(t) = u(t) \\
y(t) = B(z_1, z_2) \zeta(t)
\end{cases}
\] (6)
for some internal state variable \( \zeta \).

The polynomial \( B \) may be further decomposed into a number of factors. The main concern is the locations of zeros. Introduce for this reason the following factorizations
\[
B = b_o B_1 B_2; B_1(0, 0) = 1, B_2(1, 1) = 1.
\] (7)
The polynomial \( B_1 \) is supposed to be such that all solutions \( s = s_t \) to the equation
\[
B_1 \left( \frac{a}{s + a}, e^{-\tau s} \right) = 0
\] (8)
have \( \text{Re}(s_t) < 0 \) while \( B_2 \) contains the remaining factors. The factor \( b_o \) is a scalar, i.e., a pure constant gain. The polynomial \( B_1 \) may be called a greatest causal and stable factor of \( B \). The decomposition of \( B \) such that (7), (8) hold is always possible to do since one may always choose \( B_1 = 1 \) and all remaining factors in \( B_2 \). The problem to find a \( B_2 \) of least possible complexity will not be considered here but is of some practical importance. The reason for making this factorization is to assure internal stability in the controlled system. The factor \( B_2 \) will play the same role of system invariant as the relative degree, time delay etc. in 1-D model matching problems, see Pernebo (1981) and Aström and Wittenmark (1984).

ASSUMPTIONS
A number of restrictions on the class of control objects will now be given in order to assure solutions with good properties. Assume that
(A1) \( b_{00} = 0 \), i.e. there is no direct term from input to output.
(A2) The polynomial degrees \( n_{ij}, n_{ij}, n_{ijb}, n_{jbb} \) of \( A \) and \( B \) are all known.
(A3) \( B_2 \) is known.
(A4) \( A \) and \( B \) are coprime i.e. \( \exists M, N \in R[z_1, z_2]: AM + BN = 1 \).
(A5) The time delay \( \tau \) is known.

It should be said that the requirement to know \( B_2 \) is impractical except when \( B_2 \) may be reduced to pure powers of \( z_1 \) and \( z_2 \), i.e., low pass filters and time delays in series. The condition (A4) is formally restrictive since any two polynomials of \( R[z_1, z_2] \) have generically a common zero.

SOLUTION
The closed loop system of (2) and (3) with time-invariant \( A, B, R, S, T \) gives the transfer operator as
\[
y = \frac{BT}{AR + BS} u_t,
\] (9)
where \( R, S, T \) may be tuned to fulfil specifications.
Delay-differential systems

in terms of a reference model. This model should reflect the design trade-offs and is of the type

\[ y_m = \frac{B_mB_m}{A_m}u; \quad A_m(0,0) = 1, \]  

where \( A_m, B_m \) are desired pole- and zero-polynomials and where \( B_2 \) is the part of the B-polynomial of the control object which should not be cancelled in the closed loop system (9).

\[ P = A_mB_m; \quad T = t_0B_m; \quad t_0 = \frac{1}{b_0}, \]  

where \( b_0 \) is the constant gain introduced in (7). The model matching problem is then solved via the Diophantine equation

\[ AR + BS = P. \]  

It is necessary to obtain polynomials \( R \) and \( S \) from (12) in order to have solutions that are suitable for direct adaptive control. Such solutions do not always exist, see Example 2 below. When there are polynomial solutions \( R, S \) in two indeterminates, it is possible to express them as

\[ R(z_1,z_2) = \sum_{i=0}^{n_R} \sum_{j=0}^{n_R} r_{ij}z_1^i z_2^j; r_{00} = 1 \]
\[ S(z_1,z_2) = \sum_{i=0}^{n_S} \sum_{j=0}^{n_S} s_{ij}z_1^i z_2^j \]

for some finite numbers \( n_R, n_S, n_{R1}, n_{S1}, \) which are determined in the process of solving (12). The model matching control law (2) is then expressed explicitly as

\[ u = -\sum_{i=1}^{n_R} \sum_{j=1}^{n_S} r_{ij}z_1^i z_2^j u + \sum_{i=1}^{n_R} \sum_{j=0}^{n_S} r_{ij}z_1^i z_2^j u \]
\[ -\sum_{i=0}^{n_R} \sum_{j=0}^{n_S} s_{ij}z_1^i z_2^j + t_0(B_m u), \]

\( \psi \) is the vector \( \psi \) thus contains filtered and delayed inputs \( u \) and outputs \( y \). Then we have

\[ u(t) = -\theta^T \psi(t) \]

with

\[ \theta = (r_{10} \ldots r_{ij} \ldots s_{00} \ldots s_{ij} \ldots t_0)^T \]

and

\[ \psi = ((z_1 u) \ldots (z_1^i z_2^j u) \ldots y \ldots (z_1^i z_2^j y) \ldots - (B_m u)) \]

The estimation problem of direct adaptive control is now to calculate \( \theta \) from input–output data.

**PARAMETER ESTIMATION**

The parameters of the regulator may now be estimated from a linear model obtained by manipulations of the factorization (6) using (12) and (6), respectively.

\[ y = B_2\xi = B\frac{RA + SB}{P} \xi = \frac{B}{P}(Ru + Sy). \]

Assume that \( B_2 \) is known, cf. (A3), and define

\[ \tilde{u} = B_2\tilde{y} = B_2\tilde{y} \text{ etc.} \]

Introduce the data vector \( \varphi \)

\[ \varphi = ((z_1 \tilde{u}) \ldots (z_1 z_2 \tilde{u}) \ldots \tilde{y} \ldots (z_1^i z_2^j \tilde{y}) \ldots - (A_m \xi)) \]

in accordance with standard notation of recursive estimation, see Ljung and Söderström (1983). Then it is possible to rearrange (19) to the scalar product

\[ \tilde{u}(t) = -\theta^T \varphi(t). \]

The constant parameters \( \theta \) may now be estimated by any suitable method for identification of parameters of a linear model with known components of \( \varphi \). Recall that the estimates \( \hat{\theta} \) of the parameters \( \theta \) in (22) are intended for use in the continuous-time regulator (14). The parameters may however be estimated by discrete-time methods. One natural choice is to sample corresponding \( \tilde{u} \) and \( \varphi \) and to fit parameter estimates to the sampled data by recursive identification. For further details on recursive estimation, see Ljung and Söderström (1983).

**Remark.** The suggested methods in two operators for continuous-time systems may of course also be used for identification of "conventional" continuous-time systems by discrete-time methods. The
operator in which to formulate the parametric model for identification is then the low pass filter operator \( z_1 \). The choice of bandwidth of \( z_1 \) and the sampling period for the estimation have certainly many interesting information-theoretical aspects but these will not be elaborated on here.

This also opens possibilities of bridging the gap between the two traditions of continuous- and discrete-time identification. This is one method for on-line identification and recursive estimation of a continuous-time system by discrete-time methods while keeping the structure of the continuous-time system.

The use of the low pass filters differs from that of traditional continuous-time methods, see Eykhoff (1974), where the filter assembly is a part of the hardware for estimation. All parameter estimation is however made by discrete-time recursive methods in the approach of this paper.

**ADAPTIVE FILTERING**

An adaptive filter or “predictor” is obtained as a spin-off from the solution of the direct adaptive control problem. An observer for the quantity \( P_x \) is obtained as

\[
Ru + Sy = (RS) \left( \frac{A}{B} \right) z = P_x. \tag{23}
\]

This is seen by the following argument.

\[
Ru + Sy = (RS) \left( \frac{A}{B} \right) z = P_x. \tag{24}
\]

It is possible to carry through all of the arguments above with the purpose of deriving adaptive filters for \( P_x \) with any \( P \) such that there is a solution to (12). There is in this case no need to factorize \( B \) into \( B_1 \) and \( B_2 \) and assumptions on common zeros and prior knowledge are less restrictive.

**Example 1.** Consider the following delay-differential control object

\[
\begin{align*}
\dot{x}(t) &= -a_1 x(t) + a_2 x(t - \tau) + bu(t) \\
y(t) &= x(t),
\end{align*} \tag{25}
\]

with input \( u \) and output \( y \). The time delay \( \tau \) is supposed to be known. The problem is to design an adaptive controller

\[
u(t) = \frac{S(z_1, z_2)}{R(z_1, z_2)} y(t) + \frac{T(z_1, z_2)}{R(z_1, z_2)} u(t) \tag{26}
\]

such that the closed loop system matches the reference model

\[
y_m(t) = \frac{1}{2p + 1} u(t). \tag{27}
\]

A polynomial description of the control object in terms of the operators

\[
z_1 = \frac{1}{p + 1} \quad \text{and} \quad z_2 = e^{-p} \tag{28}
\]

is obtained via

\[
\begin{align*}
\frac{1 - z_1}{z_1} x &= -a_1 x + a_2 z_2 x + bu \\
y &= x.
\end{align*} \tag{29}
\]

This yields the transfer function

\[
j(t) = \frac{bz_1}{1 + (a_1 - 1)z_1 - a_2 z_1 z_2} u(t) = \frac{B(z_1, z_2)}{A(z_1, z_2)} u(t). \tag{30}
\]

It is here easy to see that \( A \) and \( B \) are coprime. The zero of \( B \) appears at

\[
z_1 = 0, \tag{31}
\]

where the polynomial \( A \) certainly is non-zero. The zero of (31) is—in terms of the differential operator—an infinite zero of degree 1 of the transfer function. The reference model

\[
\frac{1}{2p + 1} = \frac{0.5z_1}{1 - 0.5z_1} \frac{B_m}{A_m} \tag{32}
\]

has the same type of zero and it is therefore possible to find a solution to the pole placement problem with a regulator (26) which is causal and without derivatives. A solution with \( A_m = 1 - 0.5z_1 \) and \( B_m = 0.5 \) is obtained for

\[
\begin{align*}
R &= 1 \\
S &= \frac{1}{2} - a_1 - a_2 \tau \quad b = s_{00} + s_{01} \tau \\
T &= 1 \cdot 0.5 = t_0 B_m. \tag{33}
\end{align*}
\]

Input-matching estimation based on the linear model

\[
(z_1 u) = -(s_{00} + s_{01} \tau) (z_1 y) + t_0 (A_m y) \tag{34}
\]

or shorter

\[
\bar{u}(t) = -\theta^T \varphi^u \tag{35}
\]

with

\[
\varphi = (s_{00} s_{01} t_0)^T
\]
and

\[ \phi = ((z_1 y)(z_2 z_1 y) - (A_m y))^T. \]

Here it is possible to use identification methods for estimation of parameters of a linear model if the filtered input-output data \((z_1 u), (z_1 y), (z_1 z_2 y)\) are available.

The following simulations show the results when (34) has been sampled with sampling period \(h = 1\). See appendix for details. Recursive least-squares identification has been applied to estimate the parameters.

**Example 2.** Consider the following delay-differential equation for a control object

\[
\begin{align*}
\dot{x}(t) &= -a_1 x(t) + a_2 x(t - \tau) + bu(t - \tau) \\
y(t) &= x(t).
\end{align*}
\]

This equation differs from that of the previous example in the time delay of \(u\). The polynomial representation now becomes

\[ y(t) = \frac{b z_1 z_2}{1 + (a_1 - 1)z_1 - a_4 z_1 z_2} u(t). \]

The \(A\) and \(B\)-polynomials now both vanish at

\[ (z_1, z_2) = \left( \frac{1}{1 - a_1}, 0 \right). \]

For this reason it is not possible to solve the polynomial Diophantine equation

\[ AR + BS = P \]

for an arbitrary \(P\). It is necessary to include in \(P\) a factor with a zero at (38). The factor

\[ (1 + (a_1 - 1)z_1) \]

would be suitable when it is stable and stably invertible, see Sebek (1983). This presents no particular problem in a pure model matching problem with a known process to control. It would then be possible to include (40) as a stable common factor of both the numerator and the denominator in the reference model. This trick cannot be used in the adaptive case since the zero (38) is not known to the designer. This lack of coprimeness therefore puts a limit to the applicability of model reference adaptive control to delay-differential systems. Simulations do however show that the performance is often also very good when there is no solution to the Diophantine equation. The "least squares solutions" show slower convergence but good tracking capabilities are still exhibited.

**CONCLUSIONS**

It has been possible to formally carry over many of the algebraic results from 1-D direct adaptive control. There is however a strong practical difference between the required prior information in the two cases. The question of coprimeness is also more complicated in the 2-D case. Any solution \((z_1, z_2) = (a, b)\) to the equation

\[ A(z_1, z_2) = 0 \]

\[ B(z_1, z_2) = 0 \]

puts the constraint on \(P\) that \(P(a,b)\) must be zero in order to assure the existence of a solution to (12). In contrast to the 1-D case it is not in general possible to get rid of the common zero by pole-zero cancellation between \(A\) and \(B\), see example 2 and Morf et al. (1977).

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**REFERENCES**


APPENDIX

Details of the simulations

Example 1 involves both continuous-time and discrete-time simulation. The sampling interval is h = 1 and the time variables are called t and k, respectively. The relation is k < t < k + h between the samples.

The example requires simulation of the following equations.

Object — continuous time

\[ \frac{dx(t)}{dt} = -a_1 x(t) + a_2 x(t - r) + bu(t) \]

\[ y(t) = x(t) \]

\[ x(0) = 1. \]

Controller — continuous time

\[ u(t) = -s_0 x_0(t) - s_01 y(t - r) + 0.5 s_0 f_0(u(t)) \]

Delay \( \tau = 10 \) assumed to be known

\[ u_c(t) = \text{sign}(\sin(0.1 t)). \]

Identification model — continuous time

\[ a(t) = s_00 f_0(t) - s_01 f_0(t - r) + 0.5 f_0(t) \]

with

\[ \dot{a}(t)/dt = a(t) + u(t) \]

\[ \dot{y}(t)/dt = y(t) + y(t). \]

Identification model — sampled at times \( t = 0, 1, 2, \ldots, k, \ldots \)

\[ a(k) = s_00 f_0(k) - s_01 f_0(k - r) + 0.5 f_0(k) \]

\[ a(k) = s_00 f_0(k) - s_01 f_0(k - r) + 0.5 f_0(k) \]

\[ a(k) = a(k - 1) \]

LS estimation — discrete time

\[ \dot{\theta} = (s_00 f_0(k) - s_01 f_0(k - r))^T \psi(k) \]

\[ \psi(k) = (a(k) y(k - r) - y(k) - 0.5 f_0(k))^T \]

\[ \dot{\theta}(k) = \dot{\theta}(k - 1) + PK(\psi(k) - 0.5 f_0(k)) \]

\[ PK = \frac{1}{\lambda} \left( PK - 1 \right) \frac{1}{\lambda} \psi(k) \psi(k)^T \psi(k - 1) \psi(k) \]